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UNIVERSITY OF ALBERTA

POSITIVE DECAYING SOLUTIONS OF SEMILINEAR ELLIPTIC SYSTEMS IN $\mathbb{R}^n$

by

Odiobala Peter Odiobala

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA
SPRING, 1993

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ISBN 0-315-82194-9

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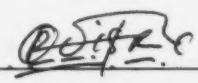

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DEDICATION

To the memory of my

Grandmother

Chief (Madam) Oyinebo T. Ebinum

ABSTRACT

In this thesis, we consider nonlinear elliptic systems of the form:

$$\ell \vec{u} = \lambda \vec{f}(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n$$

where ℓ denotes different second order uniformly elliptic operator, and $\vec{f}(x, \vec{t})$ is superlinear with subcritical growth at ∞. We show the existence of positive (componentwise) decaying solutions. Our basic tools are classical estimates of Gidas, Ni, Nirenberg and Egnell coupled with Leray-Schauder Degree Theory arguments in weighted spaces. We use also estimates of the type of Brezis and Turner. We do not assume, in general, that the system under consideration is variational, although Mountain Pass arguments are employed for certain such cases. Our approach enables us to obtain, in particular, the existence of positive solutions also for reducible systems, and the extension of several recent results, even in the scalar case.

We consider also, in this thesis, the nonpositone elliptic problem in $\mathbb{R}^n$ – that is, the problem of the existence of nontrivial solutions (possibly componentwise positive) in $\mathbb{R}^n$ to the above nonlinear equation where $\vec{f}(x, \vec{0}) \leq \vec{0}$ and $\vec{f}(x, \vec{0}) \not\equiv \vec{0}$. Existence is obtained by Mountain Pass Arguments, and we show that such a solution $\vec{u}$ is componentwise non-trivial and the sum of the component functions is positive.

In all, our methods enable us to treat nonradial coefficients and without the assumption of suitable bounds on the nonlinear terms.

ACKNOWLEDGEMENTS

I would like to express my heart-felt gratitude to my supervisor, Professor Walter Allegretto, for his patient guidance and unremitting encouragement throughout the preparation of this thesis. I indeed count myself fortunate to have worked with someone like him – with a very genuine care, interest and support in general; and unique mathematical insights in particular.

I would like to spare myself the guilt feelings that could result from missing out some important names, by simply placing on record, with appreciation, the valuable advice and support that I have received from various staff members (academic and non-academic alike) of the Department of Mathematics at this University. I have truly enjoyed the stimulating environment you all created.

I wish to thank this university – through the Department of Mathematics for the financial support I received throughout the entire period of my stay.

I would like to acknowledge my debt to my friends who stood by me all through.

Finally, I thank Mrs. Vivian Spak for her very efficient typing of the manuscript.

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CHAPTER 0

INTRODUCTION

Semilinear elliptic equations on $\mathbb{R}^n$ have attracted and continue to attract the attention of mathematicians. Indeed, the second order equation

$$\ell u \equiv -\sum D_i\big(a_{ij}(x)D_j u\big) + \sum b_j D_j u + c(x)u = f(x,u), \tag{0.1}$$

$x \in \mathbb{R}^n$ and the system form

$$\ell^k u^k \equiv -\sum D_i\big(a^k_{ij}(x)D_j u^k\big) + \sum b^k_j D_j u^k + c^k(x)u^k = f^k(x,\vec{u}), \tag{0.2}$$

$x \in \mathbb{R}^n$, $k = 1,2,\ldots,N$, $\vec{u} = (u^1,\ldots,u^N)$, have received extensive investigations in recent years. The interests on these equations have been motivated, on the one hand, by the fact that they arise naturally in many fields of study, such as physics, chemistry, biology and geometry. For example, (0.1) arises in the search for certain kinds of solitary waves, otherwise called stationary states, in the nonlinear Klein-Gordon equation:

$$\Phi_{tt} - \Delta\Phi + a^2\Phi = f(\Phi). \tag{0.3}$$

To be more precise, consider a real Klein-Gordon equation, that is $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Then, looking for a solitary wave in (0.3) of

"travelling wave" type, that is Φ of the form $\Phi(t,x) = u(x-ct)$ where $u : \mathbb{R}^n \to \mathbb{R}$, and $c \in \mathbb{R}^n$ is a fixed vector such that $|c| < 1$, leads to:

$$-\sum D_i(a_{ij}D_j u) + a^2 u = f(u) \quad \text{in} \quad \mathbb{R}^n \tag{0.4}$$

where $a_{ij} = \delta_{ij} - c_i c_j$, c_i the coordinates of c. For a complex Klein-Gordon equation, looking for a standing wave solution of (0.3), that is $\Phi(t,x) = e^{iwt}u(x)$, $w \in \mathbb{R}$, leads to the equation

$$-\Delta u + mu = f(u) \quad \text{in} \quad \mathbb{R}^n \tag{0.5}$$

where $m = a^2 - w^2$. Stationary states of a non-linear Schrödinger equation

$$i\Phi_t - \Delta\Phi = f(\Phi)$$

where $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ and f satisfies $f(\rho e^{i\theta}) = f(\rho)e^{i\theta}$, ρ, $\theta \in \mathbb{R}$, lead to a similar problem as (0.5). Systems of equations of the type (0.4) or (0.5) arise from considerations of solitary waves in coupled nonlinear Klein-Gordon equations and (or) nonlinear Schrödinger equations. Equations of the type (0.1) arise also in the study of gas dynamics - via the so-called Emden-Fowler equations; for example $-\Delta u = K(x)u^\beta$ $(\beta \geq 0)$. Emden-Fowler equations appear also in the study of fluid mechanics, relativistic mechanics and chemically reacting systems. Among other physically interesting model problems, we mention

the study of temperature distribution in an object – the steady states consideration, the Hartree-Fock theory in quantum mechanics and the "prescribing curvature problem" in Riemannian geometry. For more details and further model problems, see, for instance, H. Berestycki and P.L. Lions [BL2], D. Gogny and P.L. Lions [GL], P.L. Lions [Li3], Wong [Wo] and the references therein. On the other hand, investigations of solutions of (0.1) and (0.2) are of mathematical interest, in themselves, due to the difficulties that arise in dealing with equations in the entire space $\mathbb{R}^n$ as against a bounded domain Ω. For instance, in $\mathbb{R}^n$: L^∞ a priori estimates are not sufficient to ensure the right behavior at infinity or to avoid the trivial solution; there is the lack of compact embedding theorems of Sobolev-Kondrakov type; the Poincare Inequality no longer holds and there is the failure of the L^2-theory. It is thus the desire of many mathematicians working in this area to overcome these difficulties.

There have already been numerous investigations of the scalar equation (0.1). Among the many papers, we mention only those that have partly motivated our interest in this area: Allegretto and Yu [AY], Berestycki and Lions [BL], Chaljub-Simon and Volkmann [CV], Egnell [Eg], Gidas, Ni and Nirenberg [GNN2], Lions [Li2] and Noussair and Swanson [NS4]. For systems, the case of special interest here, for the most part, only very few papers are available – especially in the superlinear and subcritical case. Among the earliest papers, we mention those by Kawano [Ka], Kawano and Kusano [KK] and Allegretto [All]. In [Ka] and [KK], the authors considered the case $\ell^1 = \cdots = \ell^N = \ell = -\Delta$

and $N = 2$. Their nonlinearities admitted suitable radial majorants, making the barrier methods applicable. Allegretto [All], on the other hand, investigated the case $\ell^1 = \cdots = \ell^N = \ell$, $c(x) \equiv 0$ and N arbitrary. He assumed his nonlinearity to be suitably bounded and employed fixed point theorem arguments. In all the three papers, the authors established the existence of solutions in $\mathbb{R}^n$ whose components are positive, bounded and bounded away from zero. Recently, a number of authors have considered the question of existence of positive decaying (to zero) solutions in $\mathbb{R}^n$, for systems. We mention the papers by Furusho [Fur1], Kusano and Swanson [KS], and Noussair and Swanson [NS1, NS2, NS3]. Almost all of these authors looked at systems where the nonlinearities f^i were assumed suitably bounded or sublinear. Their methods included barrier methods, subdomain approximation of $\mathbb{R}^n$ and monotone iteration technique. Some of the authors considered, in addition to (0.2), the associated eigenvalue problem

$$\ell^k u^k = \lambda f^k(x, \vec{u}), \quad x \in \mathbb{R}^n, \tag{0.6}$$

with $\lambda > 0$ a constant. In particular, Noussair and Swanson [NS3] considered the system

$$-\Delta u^i + c^i(x) u^i = \lambda f^i(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n,$$

$n \geq 3$ and $0 \leq c^i(x)$ bounded. The authors considered the case of superlinear and subcritical nonlinearities with the system assumed to have variational

structure – that is, there exists a differentiable function $F(x,\vec{u})$ such that $\nabla_{\vec{u}}F(x,\vec{u}) = (f^1(x,\vec{u}),\ldots,f^N(x,\vec{u}))$. Existence results for positive solutions were obtained and conditions given on the components of $\vec{f}$ to guarantee that every such solution satisfies

$$u^i(x) = O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty, \quad i = 1,\ldots,N.$$

It seems nothing is available for nonvariational, superlinear and subcritical systems. In this thesis, we consider, for the most part, this case. We shall give conditions that guarantee the existence of positive decaying solutions of (0.6), say, under various choices of the operators ℓ^k. For existence, we shall employ topological degree and perturbation arguments since neither barrier nor variational methods are applicable. As in the references mentioned above, we shall make use of various Sobolev spaces (weighted and standard). The major difficulty with our approach arises with the task of obtaining global L^∞ a priori estimates of all positive solutions of the equations under consideration. In fact it does not appear possible at the moment to obtain such estimates without placing some restrictions on the nonlinear terms $f^i(x,\vec{u})$, say, in the x-variable.

Complementary results will be obtained for variational systems by the Mountain Pass arguments – a method we shall also use to handle the existence question for the nonpositone problem – $f^i(x,\vec{0}) \leq 0$ with $f^i(x,\vec{0}) \not\equiv 0$ for $x \in \mathbb{R}^n$ and $1 \leq i \leq N$. Such a problem started receiving attention after it

appeared in a series of papers by Smoller and Wasserman [SW1, SW2, SW3] on symmetry – breaking for positive solutions of semilinear elliptic equations. Other papers that have investigated nonpositone problems are those by Allegretto, Nistri and Zecca [ANZ], Brown, Castro and Shivaji [BCS], Brown and Shivaji [BSh], Castro and Shivaji [CS1], [CS2] and finally we mention the thesis of Unsurangsie [Un]. The prototype equation considered is of the following type:

$$(0.7) \qquad \begin{cases} -\sum D_i\big(a_{ij}(x)D_j u\big) = \lambda f(x,u) & \text{in} \quad \Omega \\ u \equiv 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $\lambda > 0$, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain and $n \geq 3$. The problem is to obtain conditions for the existence of positive solutions – starting out with the assumption: $f(x,0) \leq 0$, $f(x,0) \not\equiv 0$. To the best of our knowledge, all the previous considerations have dealt with the case of a bounded domain Ω and scalar equations. In this thesis, we extend some of the known results to decaying solutions in the entire space $\mathbb{R}^n$ and to systems of equations, although we obtain somewhat weaker results for the nonpositone systems.

This thesis is organised as follows: In Chapter 1, we consider the problem (0.6) with $\ell^1 = \cdots = \ell^N = -\Delta$. We establish various conditions for (0.6) to have positive decaying solutions under the assumptions that the system is non-variational and that the nonlinearities $f^i(x,\vec{u})$ are superlinear and subcritical. We present our results, first for any $\lambda > 0$ and then for $\lambda > 0$, sufficiently

small. In the latter case, the system may also have nonnegative nontrivial solutions. In Chapter 2, we modify slightly the arguments in Chapter 1 to treat the case in which the operators in (0.6) are now taken as $\ell^1 = \cdots = \ell^N = -\Delta + 1$; the system continues to be nonvariational. Allowing the elliptic operators ℓ^k to admit first order terms is the object of our study in Chapter 3. However, we are only able to cover a smaller growth range in this chapter than in the first two chapters. In Chapter 4, we consider a variational system and establish conditions for the system to admit a positive decaying solution. Here the existence is by Mountain Pass arguments. We again use Mountain Pass arguments to investigate the nonpositone problems in Chapter 5. The thesis concludes, in Chapter 6, with remarks and a consideration of some outstanding problems.

Let us conclude this introductory chapter by setting down the following notation. We set $D_i u = \frac{\partial u}{\partial x_i} = u_{x_i}$ where u is a differentiable function, and we write $\|u\|_{L^p} = \|u\|_{L^p(\mathbb{R}^n)} = \|u\|_p =$ the norm of $u \in L^p(\mathbb{R}^n)$ interchangeably throughout this thesis. Finally vector inequalities will be understood to hold componentwise – for example, $\vec{u} > \vec{0}$ (resp. $\vec{u} \geq \vec{0}$) will mean that all the components of $\vec{u}$ are positive (resp. nonnegative). Functions introduced are assumed smooth in their arguments unless otherwise specified.

We present explicitly results for $n \geq 3$. The cases where $n = 1, 2$ can usually be treated in the same way and analogous – often stronger – results hold.

CHAPTER 1

DECAYING SOLUTIONS OF ELLIPTIC SYSTEMS

1.1. Introduction

Our interest in this chapter is the study of positive decaying solutions for second order semilinear elliptic systems of the type

$$-\Delta u^i = \lambda f^i(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n, \quad n > 2,\ 1 \le i \le N, \tag{1.1.1}$$

where $\lambda > 0$ is a constant, $x = (x_1, \ldots, x_n)$, $\vec{u} = (u^1, u^2, \ldots, u^N)$ and Δ denotes the n-dimensional Laplacian. The functions $f^i(x, \vec{u})$ are assumed to be superlinear and subcritical, i.e.; $|f^i(x, \vec{u})|$ grows faster than linearly in $|\vec{u}|$ and $|f^i(x, \vec{u})| \le C|\vec{u}|^\beta$ for $|\vec{u}|$ large, $1 < \beta < \frac{n+2}{n-2}$. More precise conditions on f^i will be given in the sections that follow.

The question of existence, non-existence and qualitative properties of positive solutions for semilinear elliptic systems has received only very little attention – especially in the superlinear case. Among the earliest papers, we mention those by H. Amann [Am], R. Nussbaum [Nu] and W. Troy [Tr]. H. Amann studied the existence question for sublinear systems. He indeed developed a reasonably satisfactory existence theory for this class of systems. R. Nussbaum, on the other hand, investigated superlinear systems. For his existence results, he required the nonlinear term to have a growth of order $|\vec{u}|^\beta$, $\beta < \frac{n}{n-1}$ or $\frac{n}{n-2}$ – depending on the boundary condition. W. Troy obtained some symmetry properties of positive solutions for elliptic systems. All the above investigations

were for bounded domains. Among the other papers for bounded domains that appeared after the survey article of P.L. Lions [Li1] – where the question of existence of positive solutions for superlinear elliptic systems was considered an open problem – are those by C. Cosner [Co] and Qing Jie [Qi]. In [Co], Cosner obtained existence results for superlinear elliptic systems with autonomous nonlinearity. He assumed growths of order $|\vec{u}|^\beta$, $\beta < \frac{n+1}{n-1}$ or $\frac{n}{n-2}$ – depending on the nature of the operator considered. In [Qi], Qing Jie achieved a growth of order $|\vec{u}|^\beta$, $\beta < \frac{n+2}{n-2}$. This growth condition is considered, in general, the 'best possible' in order to expect existence – a fact due to the result of Pohozaev [Po]. It was observed by Cosner [Co] that this growth condition is always achieved if the system under consideration is assumed to have variational structure; that is, if the operators are all of variational form and the nonlinearities $f^i(x,\vec{u})$ are such that there exists a differentiable function $F(x,\vec{u})$ with $\nabla_{\vec{u}} F(x,\vec{u}) = \big(f^1(x,\vec{u}), f^2(x,\vec{u}), \ldots, f^N(x,\vec{u})\big)$. This observation is due to the applicability of the abstract results of Ambrosetti and Rabinowitz [AR] in this case. The situation in the case of the Entire Space $\mathbb{R}^n$, is still in its early stage of comprehension. Some of the earliest results obtained in this case are those by N. Kawano [Ka], N. Kawano and T. Kusano [KK] and W. Allegretto [All]. In [Ka] and [KK], the authors considered the case $\ell = -\Delta$ and $N = 2$. They assumed their nonlinearities to admit suitable radial majorants which enabled them to construct spherically symmetric supersolutions and subsolutions, thus making the barrier methods applicable. W. Allegretto [All], on

the other hand, investigated the case of a more general elliptic expression

$$\ell\vec{u} = -\sum_{i,j=1}^{n} D_i\big(a_{ij}(x)D_j\vec{u}\big) + \sum_{j=1}^{n} b_j D_j\vec{u}$$

with N taken to be arbitrary. He assumed his nonlinearity to be suitably bounded and employed fixed point theorem arguments. In all the three papers, the authors established the existence of solutions in $\mathbb{R}^n$ whose components are positive, bounded and bounded away from zero. It seems that only very little is known about the existence of positive decaying solutions in $\mathbb{R}^n$, for systems. Recent results in this direction are those obtained by Y. Furusho [Fur1], E.S. Noussair and C.A. Swanson [NS1], [NS3]. In [NS1], the authors considered the case that the nonlinearities are suitably bounded. They gave conditions which guarantee the existence of positive decaying solutions, with the elliptic operators given by $\ell^K = -\sum_{i,j=1}^{n} D_i[a_{ij}(x)D_j] + b^K(x)$. Their methods consisted of subdomain approximation of $\mathbb{R}^n$ and monotone iteration technique. Y. Furusho in [Fur1] looked at systems where the nonlinearities f^i were in addition allowed to depend on the gradient of the u^i's. He considered the most general uniformly elliptic operators and used what he called a new barrier method to obtain existence result for positive decaying solution in the case that each f^i is a sublinear function of u^i at $u^i = 0$. In [NS3], Noussair and Swanson considered the system

$$-\Delta u^i + a^i(x)u^i = \lambda f^i(x,\vec{u}) \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3,$$

with $a^i(x) \geq 0$ and bounded. Although they considered the superlinear and subcritical nonlinearities, they assumed that their system has variational structure. This is a strong requirement which will not be satisfied in many cases. Existence results for positive solutions were given in the case that $f^i(x, \vec{u})$ is assumed to satisfy a power growth condition in the components of $\vec{u}$, with exponents $\beta = (\beta^1, \ldots, \beta^N)$ of each power of $\vec{u}$ restricted by $1 < |\beta| < \frac{n+2}{n-2}$. Here $|\beta| = \sum_{i=1}^{N} \beta^i$ for the multi-index β. Conditions were given on the f^i's which guarantee that every such solution satisfies $u^i(x) = O(|x|^{2-n})$ as $|x| \to \infty$, $i = 1, 2, \ldots, N$. Among other recent papers for systems in $\mathbb{R}^n$, we mention those by E.S. Noussair and C.A. Swanson [NS2], and T. Kusano and C.A. Swanson [KS], where barrier methods were employed.

Our main objective in this chapter is to establish existence of positive (componentwise) solutions $\vec{u}$ to (1.1.1) such that $\vec{u} \to \vec{0}$ at ∞. As stated above, our interest is in the case of superlinear and subcritical nonlinearities and we shall cover the full subcritical growth range. We treat the nonvariational case, that is, we do not assume that our system has variational structure. We also do not deal with bounded nonlinearities as in [Al1], [NS1] and do not assume that our nonlinearities admit suitable radial majorants as in [Ka], [KK] and [NS2]. Our method involves the use of certain monotonicity results to obtain global a priori estimates for positive solutions of (1.1.1) and then we employ topological degree arguments via a Krasnosel'skii's type theorem and certain perturbation arguments – to obtain our existence results. This method

has the disadvantage of requiring the elliptic operators not to have first-order terms, and the nonlinearities to satisfy certain monotonicity conditions.

The plan of this chapter is as follows: after this introduction, we proceed in Section 1.2, to present some useful background materials – which will include the introduction of the Banach spaces that we shall work with. In Section 1.3, we present our first existence result. We prove in particular that if the nonlinearities f^i are assumed to satisfy, among others, certain monotonicity conditions in the x variable, then (1.1.1) has a positive decaying solution for any $\lambda > 0$. In Section 1.4, we relax somewhat this monotonicity assumption and employ perturbation arguments to obtain existence results – this time, only for $\lambda > 0$ small enough. Finally in Section 1.5, we conclude the chapter with illustrative examples.

1.2. Background Materials

In this section, we present certain useful background materials. By E we denote the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm:

$$\|V\|_E^2 = \int_{\mathbb{R}^n} |\nabla V|^2 dx. \tag{1.2.1}$$

We recall the following well known inequality of Hardy, which holds if $n \geq 3$:

$$\int_{\mathbb{R}^n} \frac{|V(x)|^2}{1+|x|^2}\, dx \leq C_2 \int_{\mathbb{R}^n} |\nabla V(x)|^2 dx \tag{1.2.2}$$

where C_2 is a constant independent of $V \in C_0^\infty(\mathbb{R}^n)$. By virtue of this inequality, we observe that $E \sim \overline{E}$ where $\overline{E}$ is the Weighted Space obtained as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|\omega\|_{\overline{E}} = \int_{\mathbb{R}^n} (|\nabla\omega|^2 + \sigma\omega^2)dx \tag{1.2.3}$$

where $\sigma(x) = (1+|x|^2)^{-1}$. We consider only solutions to (1.1.1) in E. The various norms of the vector functions $\vec{u}$ are defined in terms of the norms of the components of $\vec{u}$ in obvious ways. For example,

$$\|\vec{u}\|_E^2 = \sum_{i=1}^N \|u^i\|_E^2. \tag{1.2.4}$$

If $\vec{u} > \vec{0}$ then $|\vec{u}|$ will denote $\sum_{i=1}^N u^i$; but in general, $|\vec{u}| = \sum_{i=1}^N |u^i|$. We recall the following properties of E.

LEMMA 1.0.

(a) Sobolev's Inequality: $\|u\|_{\frac{2n}{n-2}} \le C\|\nabla u\|_2$ *holds in* E.

(b) If $u \in E$ *then* $u^+, u^- \in E$.

The proof of Lemma 1.0 can be found, for example, in [AY].

We recall here, also, the notion of quasi-irreducibility of the nonlinear terms, introduced by C. Cosner [Co]. We say that $\vec{f}(x,\vec{u})$ is quasi-irreducible if it is such that every non-trivial non-negative vector solution of (1.1.1), say, is componentwise positive. Cosner stated as an example the following condition:

$\vec{f}$ is such that if $u^i > 0$ for $i \in \Gamma \subsetneq \{1, \ldots, N\}$, with $u^k \geq 0$ for all k, then $f^j(x, \vec{u}) > 0$ for some $j \notin \Gamma$ and some x.

We shall obtain our existence result in this chapter via a topological degree argument based on a fixed point result due to M.A. Krasnosel'skii. The statement of this abstract result, given below, is that found, for example, in Figueiredo, Lions and Nussbaum [FLN].

First we recall that $i_C(\Phi, W)$ denotes the fixed point index of Φ on W. By definition,

$$i_C(\Phi, W) = \deg(I - \varphi, \theta, 0)$$

where θ is a bounded, open subset of a Banach space Y with $\theta \cap C = W$; and $\varphi : \overline{\theta} \to C$ is a compact, continuous map such that $\varphi|_{\overline{W}} = \Phi$. $\deg(I - \varphi, \theta, 0)$ is the degree of the mapping $I - \varphi$ relative to the point 0 and the set θ. C is a closed, convex subset of Y.

PROPOSITION E. *Let C be a cone in a Banach space Y and $\Phi : C \to C$ a compact map such that $\Phi(0) = 0$. Assume that there exists numbers $0 < r < R$ and a vector $\widehat{v} \in C \setminus \{0\}$ such that*

(i) $x \neq t\Phi(x)$ for $0 \leq t \leq 1$ and $\|x\| = r$, and

(ii) $x \neq \Phi(x) + t\widehat{v}$ for $t \geq 0$ and $\|x\| = R$.

Then if

$$U = \{x \in C : r < \|x\| < R\}$$

and $B_\rho = \{x \in C : \|x\| < \rho\}$. *One has:*

$$i_C(\Phi, B_R) = 0, \quad i_C(\Phi, B_r) = 1 \quad \textit{and} \quad i_C(\Phi, U) = -1.$$

In particular, Φ *has a fixed point in* U.

We assume throughout this chapter that all the functions f^i are Hölder continuous with

(1.2.5) $\qquad \vec{f}(x, \vec{\xi}) \geq \vec{0}$ for $\vec{\xi} \geq 0$, $\vec{\xi} \neq \vec{0}$ and $\vec{f}(x, \vec{0}) = \vec{0}$;

(1.2.6) $\qquad \vec{f}$ is nondecreasing in $\vec{\xi} \geq 0$.

Various explicit growth and monotonicity assumptions will be given below in each section.

We recall that vector inequalities will be understood to hold componentwise; for example $\vec{u} > \vec{0}$ (resp. $\vec{u} \geq 0$) will mean that all the components of $\vec{u}$ are positive (resp. nonnegative).

1.3. Positive Solutions for all $\lambda > 0$

We consider in this section conditions under which equation (1.1.1) has positive solutions for all $\lambda > 0$. Our assumptions on $\vec{f}$ are as follows:

(I-1) There exists $\widetilde{\mu} > 0$ such that if $\mu > \widetilde{\mu}$ then for any $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, n\}$ and $\vec{\xi} \geq 0$ we have:

$$f^i(x, \vec{\xi}) \geq (\not\equiv)\, f^i(x^{\mu_j}, \vec{\xi}) \quad \text{if} \quad x_j < (x^{\mu_j})_j$$

and

$$f^i(x,\vec{\xi}) \geq (\not\equiv)\, f^i(x^{-\mu_j},\vec{\xi}) \quad \text{if} \quad x_j > (x^{-\mu_j})_j.$$

Here x^{μ_j} denotes the reflection of x about the plane $x_j = \mu$, [GNN2].

(I-2) There exists a continuous, bounded function $P(x) > 0$ such that

$$\lim_{\substack{|\vec{\xi}|\to\infty \\ \vec{\xi}\geq\vec{0}}} \frac{|\vec{f}(x,\vec{\xi})|}{|\vec{\xi}|^\gamma} = P(x)$$

locally uniformly in x.

(I-3) There exists a function $C(x) \in L^p \cap L^\infty$, $C(x) \leq K|x|^{-\beta}$ at ∞ such that $|\vec{f}(x,\vec{\xi})| \leq C(x)|\vec{\xi}|^\gamma$ for $\vec{\xi} \geq 0$, for some $p < \frac{n}{2}$ and $\beta + \gamma(n-2) > (n+1)$, $1 < \gamma < \frac{(n+2)}{(n-2)}$.

(I-4) $\vec{f}$ is quasi-irreducible.

REMARK: (a) We note that in assumption (I-3), p is independent of γ; and if $\gamma(n-2) > n+1$, then we may take $\beta = 0$. If $\gamma < \frac{n}{n-2}$, many of these conditions can be dropped altogether as we show in Chapter 3. Consequently $\beta \neq 0$ only for $\frac{n}{n-2} \leq \gamma \leq \frac{n+1}{n-2}$.

(b) Assumptions (I-1) - (I-4) are quite restrictive, but the results obtained in this section, under these assumptions, will furnish the basis for those obtained in the next section. Examples of functions $\vec{f}$ for which (I-1) - (I-4) hold will be explicitly given later but we observe that if $\vec{f}$ is radial in x and $\vec{f}' \leq \vec{0}$

for $|x|$ near ∞ then (I-1) holds. We also note that if (I-1) holds for $\vec{f}$ then it also holds for $\vec{f}+\vec{g}$ where $g^i = \varphi^i(x)(\xi^i)^\gamma$ with $\varphi^i \in C_0^\infty$, $\varphi^i \geq 0$. Finally, we note that $P(x) \leq C(x)$.

We begin with the following proposition:

PROPOSITION 1.1. *Let* $u^i \in E$ $(1 \leq i \leq N)$ *be a nonnegative solution of (1.1.1) such that* $\vec{u} \not\equiv \vec{0}$, *and let* f^i $(1 \leq i \leq N)$ *satisfy conditions (1.2.5) and (I-3). Then*

$$\lim_{|x|\to\infty} |x|^{n-2}|\vec{u}(x)| = K$$

where K *is a finite positive number.*

PROOF: The proof essentially follows that of Theorem 2 of Egnell [Eg] by first observing the fact that $-\Delta(|\vec{u}|) = \lambda|\vec{f}(x,\vec{u})|$ and also, that if $C(x) \in L^p \cap L^\infty$ with $p < \frac{n}{2}$, then $C(x) \in L^q$ with $q = \frac{2n}{[n+2-(n-2)\gamma]}$.

The next proposition gives asymptotic results near infinity in the components of a solution $\vec{u}$ of (1.1.1).

PROPOSITION 1.2. *Let* $\vec{u} > \vec{0}$ *solve (1.1.1) with* $\vec{f}$ *satisfying the conditions of Proposition 1.1; then for each* $i = 1, \ldots, N$ *and* $j = 1, \ldots, n$

(a) $\lim_{|x|\to\infty} |x|^{n-2}u^i = C\int_{\mathbb{R}^n} f^i(y,\vec{u}(y))dy > 0$;

(b) $\frac{|x|^n}{x_j}\frac{\partial u^i}{\partial x_j} \to -(n-2)C\int_{\mathbb{R}^n} f^i(y,\vec{u}(y))dy$ as $|x| \to \infty$ with $x_j \to \infty$;

(c) If $\mu^k \in \mathbb{R} \to \mu$ *and* $\{x^k\} \to \infty$ *with* $x_j^k < \mu^k$ *then*

$$\frac{|x^k|^n}{\mu^k - x_j^k}\left(u^i(x^k) - u^i((x^k)^{\mu_j^k})\right) \to 2(n-2)C\int_{\mathbb{R}^n} f^i(y, \vec{u}(y))(\mu - y_j)$$

where x^{μ_j} *is as defined in (I-1).*

PROOF: By Proposition 1.1, we have $|\vec{u}| \sim |x|^{2-n}$ at ∞. It follows then from assumptions that $f^i(x, \vec{u}) = 0(|x|^{-q})$ near infinity for some $q > n+1$, and the proof of the proposition is now identical with that of Lemma 2.1 in [GNN2].

The following theorem contains our monotonicity results.

THEOREM 1.3. *Let* f^i $(1 \le i \le N)$ *satisfy conditions (1.2.3), (1.2.4) and (I-1) - (I-3), and* $\vec{u} > \vec{0}$ *solve (1.1.1). We have:*

(a) For any $j \in \{1, \ldots, n\}$, *there exists* $\mu_0 > 0$ *such that for all* $\mu \ge \mu_0$,

(1.3.1) $\quad u^i(x) > u^i(x^{\mu_j})$ *if* $x_j < \mu$ *and* $u^i(x) > u^i(x^{-\mu_j})$ *if* $x_j > -\mu$;

(b) If $|x_j| > \widetilde{\mu}$, *then*

(1.3.2) $$x_j \frac{\partial u^i(x)}{\partial x_j} < 0;$$

where $\widetilde{\mu}$ *is as in (I-1).*

Before we give the proof of Theorem 1.3 we present the following useful lemma.

LEMMA 1.4. *Let* $u^i > 0 \quad (1 \le i \le N)$ *solve (1.1.1) with* $f^i \quad (1 \le i \le N)$ *satisfying (1.2.5), (1.2.6) and (I-1). Assume that for some* $\mu > \widetilde{\mu}$ *and for all* $i \quad (1 \le i \le N)$ *the function* u^i *satisfies*

$$u^i(x) \ge u^i(x^{\mu_1}), \quad u^i(x) \not\equiv u^i(x^{\mu_1}) \quad \text{for} \quad x_1 < \mu. \tag{1.3.3}$$

Then

$$u^i(x) > u^i(x^{\mu_1}) \quad \text{if} \quad x_1 < \mu \quad \text{and}$$

$$u^i_{x_1}(x) < 0 \quad \text{on} \quad T_\mu,$$

where T_μ *is the plane* $x_1 = \mu$.

PROOF: Follows essentially the arguments in [GNN1]. We observe that we do not need a systems form of the maximum principle for this, since the properties of $\vec{f}$ suffice.

Indeed if we set

$$\vec{v}(x) = \vec{u}(x^{\mu_1}) \quad \text{then}$$

$$-\Delta(\vec{u} - \vec{v}) = \lambda[\vec{f}(x, \vec{u}) - \vec{f}(x^{\mu_1}, \vec{v})]$$

$$= \lambda[\vec{f}(x,\vec{u}) - \vec{f}(x^{\mu_1},\vec{u}) + \vec{f}(x^{\mu_1},\vec{u}) - \vec{f}(x^{\mu_1},\vec{v})]$$

$$\geq \vec{0}.$$

PROOF OF THEOREM 1.3: By using Proposition 1.2(c), we argue as in Lemma 2.2 in [GNN2] to obtain (a). As far as part (b) is concerned, we set

$$\overline{\mu}_1 = \inf\{\mu | u^i(x) > u^i(x^{\mu_1}) \quad \text{for} \quad i = 1,\ldots,N, \quad x_1 < \mu\}.$$

Using Lemma 1.4 and arguing as in [GNN2], we note that if condition (1.3.1) holds for some $\mu > \widetilde{\mu}$, then it holds for all $\widehat{\mu}$ in a neighborhood of μ. Without loss of generality, we assume that $\mu_0 > \widetilde{\mu}$ and thus we see that $\overline{\mu}_1$ exists. Again as in [GNN2] and in view of assumption (I-1), we conclude that if $\overline{\mu}_1 > \widetilde{\mu}$, then $\overline{\mu}_1$ cannot be the infimum. Assumption (I-1) shows that the same arguments hold with the positive x_1 axis reversed, and that they may be repeated for each $j = 1,\ldots,n$ – which proves part (b) and the theorem is established.

By combining Theorem 1.3 with estimates of the type of Gidas and Spruck [GS], we obtain a global L^∞ a priori estimate for all positive solutions of (1.1.1). We have:

PROPOSITION 1.5. *Under the hypotheses of Theorem 1.3, there exists a ball B and constant K such that for any positive solution $\vec{u}$ of (1.1.1) we have that $|\vec{u}|$ takes on its maximum in B and $\| |\vec{u}| \|_{L^\infty} \leq K$ in $\mathbb{R}^n$.*

PROOF: From Theorem 1.3(b), we have that if $|x_j| > \tilde{\mu}$ then $x_j \frac{\partial u^i}{\partial x_j} \leq 0$, $j = 1, \ldots, n$. It follows that there exists a ball B such that the maximum of $|\vec{u}|$ occurs in B. Indeed, we may take $B = \{x \in \mathbb{R}^n \mid |x| < R\}$ with $R \gg \tilde{\mu}$ chosen so that the maximum occurs in the interior of B. It thus suffices to show that the solution $\vec{u}$ is such that $|\vec{u}|$ is L^∞ a priori bounded in B. Suppose there exists a sequence $\{x^k\}$ such that $M_k = \sup_k |\vec{u}_k| = |\vec{u}_k(x^k)| \to \infty$ as $k \to \infty$. Without loss of generality there exists a finite x^0 such that $x^k \to x^0$. In view of assumption (I-2), we see that $P(x^0) > 0$. Let k be fixed. Define a diffeomorphism of $\mathbb{R}^n$ by

$$F_k : x \to y$$

$$x = x^k + M_k^{-\frac{(\gamma-1)}{2}} y,$$

and new functions V_k^i by

$$u_k^i = M_k V_k^i \circ F_k.$$

Let $d > 0$ be fixed, and $\frac{B_d}{\beta_k}(0)$ be the ball centered at 0, with radius equal to $\frac{d}{\beta_k}$, where $\beta_k = M_k^{-\frac{(\gamma-1)}{2}}$. We have

$$\begin{aligned} \sup_{|y| < \frac{d}{\beta_k}} |\vec{V}_k(y)| &= \sup_{|x - x^k| < d} |\vec{V}_k| \circ F_k(x) \\ &= M_k^{-1} \sup_{|x - x^k| < d} |\vec{u}_k(x)| = 1. \end{aligned}$$

Noting that $-\Delta(|\vec{u}|) = \lambda |\vec{f}(x, \vec{u})|$, we may apply the procedure of Gidas and Spruck [GS] to obtain a contradiction.

Let us now observe the following elementary estimate indicating the equivalence of the L^∞ and $\|\cdot\|_E$ norms for positive solutions of (1.1.1).

LEMMA 1.6. *Let* $\vec{u} > \vec{0}$ *solve (1.1.1) with* $\vec{f}$ *satisfying the above stated conditions. Then there exists continuous, bounded and positive functions* a, b *such that*

$$\| |\vec{u}| \|_{L^\infty} \leq a(\| |\vec{u}| \|_E) \leq a(K'\|\vec{u}\|_E), \tag{1.3.4}$$

and

$$\|\vec{u}\|_E \leq b(\| |\vec{u}| \|_\infty). \tag{1.3.5}$$

PROOF: First, we observe that $-\Delta(|\vec{u}|) = \lambda \sum f^i(x, \vec{u})$ implies (1.3.4) by well-known scalar bootstrap arguments, [AY], in view of assumption (I-3). To see (1.3.5), we observe that

$$\begin{aligned}
\|\vec{u}\|_E^2 &\leq K_1 \int C(x)|\vec{u}|^{2-\varepsilon} \cdot |\vec{u}|^{\gamma-1+\varepsilon} \\
&\leq K_2 \| |\vec{u}| \|_{L^\infty}^{\gamma-1+\varepsilon} \| |\vec{u}| \|_E^{2-\varepsilon} \cdot \|C\|_{\frac{2n}{4+\varepsilon(n-2)}} \\
&\leq K_2 \| |\vec{u}| \|_{L^\infty}^{\gamma-1+\varepsilon} \|\vec{u}\|_E^{2-\varepsilon} \cdot \|C\|_{\frac{2n}{4+\varepsilon(n-2)}} .
\end{aligned}$$

Choosing ε sufficiently small, yields

$$\|\vec{u}\|_E \leq K \| |\vec{u}| \|_{L^\infty}^{\frac{\gamma-1+\varepsilon}{\varepsilon}} .$$

As an immediate consequence we have:

THEOREM 1.7. *Let* $\Phi(\vec{u}) = (-\Delta)^{-1}[\vec{f}^*(x,\vec{u})]$ *with* $\vec{f}^*(x,\vec{u}) = \vec{f}(x,\vec{u}^+)$. *Then* Φ *is a continuous and compact map from* E *to* E *and satisfies the conditions of Proposition E.*

PROOF: We first observe that Φ maps E to E. Indeed, for any $\vec{u} \in E$, we observe that the Nemiskii operator $H : E \to E$ given by $H(\vec{u})(x) = \vec{f}^*(x,\vec{u})$ induces a linear functional on E by the relation $H(\vec{u})(\vec{\varphi}) = \int_{\mathbb{R}^n} \vec{f}(x,\vec{u})\cdot\vec{\varphi}dx$ for $\vec{\varphi} \in E$. By arguing as in the scalar case (see for example [AY]), we have that the operator H is continuous and compact by using assumption (I-3). The continuity and compactness of the operator Φ then follows from the properties of $(-\Delta)^{-1}$. Next, we show that Φ satisfies the conditions of Proposition E. $\Phi(\vec{0}) = \vec{0}$ obviously holds. It then suffices to show that conditions (i) and (ii) of Propostion E are satisfied. Now (i) is immediate, since if

$$-\Delta\vec{u} = s\vec{f}^*(x,\vec{u}) = s\vec{f}(x,\vec{u}^+),$$

$0 \le s \le 1$, then $\vec{u} \ge \vec{0}$ with $\vec{u} \not\equiv \vec{0}$ as $\vec{u}$ is nontrivial. Whence

$$\begin{aligned} -\Delta(|\vec{u}|) &\le sC(x)(|\vec{u}|)^\gamma \\ &\le s\| \, |\vec{u}| \, \|_{L^\infty}^{\gamma-1} C(x)|\vec{u}|. \end{aligned}$$

We observe that there exists an eigenfunction $J > 0,\ J \in E$ such that $-\Delta J = \lambda C(x)J$ [Al2]. That is,

$$\lambda \int C(x)J|\vec{u}| \leq s\| \ |\vec{u}| \ \|_{L^\infty}^{\gamma-1} \int C(x)J|\vec{u}|,$$

whence $\| \ |\vec{u}| \ \|_{L^\infty}^{\gamma-1} \geq \frac{\lambda}{s} \geq \lambda$ and by Lemma 1.6, $\|\vec{u}\|_E \geq C$. We may thus pick $0 < r < C(\lambda, a)$ where a denote the function in Lemma 1.6. As for condition (ii), we recall that it suffices to consider

$$-\Delta\vec{u} = \vec{f}^*(x,\vec{u}) + tJI \tag{1.3.6}$$

where J is now the eigenfunction for the first eigenvalue to the Dirichlet problem $-\Delta$ in a ball B (extended to be zero outside B), and I denotes the identity matrix. We first observe that t must be bounded for a solution to exist. Indeed if $\vec{u}$ solves (1.3.6) with $t > 0$, then $\vec{u} \geq 0,\ \vec{u} \not\equiv 0$ and $\vec{f}^*(x,\vec{u}) = \vec{f}(x,\vec{u})$. Now

$$-\Delta(|\vec{u}|) = \sum f^i(x,\vec{u}) + NtJ.$$

Since $\frac{\sum_i f^i(x,\vec{\xi})}{|\vec{\xi}|^\gamma} \to P(x)$ uniformly in B, we have for $|\vec{\xi}|$ large enough, say $|\vec{\xi}| > a$,

$$\frac{\sum_i f^i(x,\vec{\xi})}{|\vec{\xi}|^\gamma} \geq \frac{P(x)}{2} \quad \text{in} \quad B$$

while

$$\frac{\left|\sum_i f^i(x,\vec{\xi}) - \frac{P(x)}{2}\,|\vec{\xi}|^\gamma\right|}{|\vec{\xi}|} \le K$$

for $|\vec{\xi}| \le a$ and some constant K. We thus conclude that

$$\sum_i f^i(x,\vec{\xi}) \ge \frac{P(x)}{2}\,|\vec{\xi}|^\gamma - K|\vec{\xi}|$$

and hence

$$-\Delta(|\vec{u}|) + K|\vec{u}| \ge \frac{P(x)}{2}\,|\vec{u}|^\gamma + NtJ \quad \text{in} \quad B$$

with $|\vec{u}| > C > 0$ in B for some C. On the other hand, if ω solves

$$-\Delta(\omega) + K\omega = \varepsilon J \quad \text{in} \quad B$$

$$\omega = 0 \quad \text{on} \quad \partial B$$

for $\varepsilon > 0$ small, then $0 < \omega < C$ in B and

$$-\Delta(\omega) + K\omega \le \frac{P(x)}{2}\,\omega^\gamma + NtJ.$$

We conclude that ω, $|\vec{u}|$ form an ordered lower-upper solution pair, and thus the existence of a solution $V > 0$ to the problem

$$-\Delta V + KV = \frac{P(x)}{2}\,V^\gamma + NtJ \quad \text{in} \quad B,$$

$$V = 0 \quad \text{on} \quad \partial B.$$

Following Brezis and Turner [BT] we conclude that Nt is bounded. We may now apply Proposition 1.5 and Lemma 1.6 to obtain $\|\vec{u}\|_E \leq K$ for some K, since J has compact support; and (ii) follows with $\widehat{V} = NJI$ (I = identity matrix). We note that if $t = 0$ then $a(0) = 0$ and $a(\xi) > 0$ if $\xi > 0$, where $a(\xi)$ denotes the function appearing in Lemma 1.6.

COROLLARY 1.8. *System (1.1.1) has a positive solution.*

PROOF: We have $-\Delta(\vec{u}) = \vec{f}(x, \vec{u}^+) \geq 0$ by Theorem 1.7. Therefore $\vec{u} \geq \vec{0}$, $\vec{u} \not\equiv \vec{0}$ as $|\vec{u}| \neq \vec{0}$ by the construction. We conclude that $\vec{u} > \vec{0}$ by assumption (I-4).

Finally for such solutions in E we have the following:

LEMMA 1.9. *(a)* $\vec{u} \in L^p(\mathbb{R}^n)$, $\frac{2n}{n-2} \leq p \leq \infty$.

(b) $\lim_{|x| \to \infty} u^i = 0$ *for* $i = 1, \ldots, N$.

PROOF: We can reduce the situation to an analogue of scalar results as follows. We set $\vec{u}_k(x) = \left(u_k^1(x), u_k^2(x), \ldots, u_k^N(x)\right)$ where $u_k^i(x) = \min\{u^i(x), k\}$, $i = 1, \ldots, N$ and $k = 1, 2, \ldots$. For any real number $m \geq 1$, let

$$[\vec{u}_k(x)]^m = \left([u_k^1(x)]^m, \ldots, [u_k^N(x)]^m\right).$$

Taking $\vec{\varphi}(x) = [\vec{u}_k(x)]^m$, we have, with $\langle \cdot, \cdot \rangle$ denoting the E inner product,

$$\langle \vec{u}, [\vec{u}_k(x)]^m \rangle \leq \lambda \int_{\mathbb{R}^n} \sum_{j=1}^{N} |f^j(x, \vec{u})| \cdot [u_k^j]^m dx$$

$$\leq \sum_{j=1}^{N} \lambda \int_{\mathbb{R}^n} |f^j(x, \vec{u})| \cdot [u^j]^m dx.$$

But

$$\langle \vec{u}, [\vec{u}_k(x)]^m \rangle \geq \sum_{j=1}^{N} \frac{4m}{(m+1)^2} \| [u_k^j]^{\frac{(m+1)}{2}} \|^2_{-\Delta}$$

$$\geq \sum_{j=1}^{N} C(n,m) [\| u_k^j \|_{\frac{(m+1)n}{n-2}}]^{m+1}.$$

Thus

$$C \| \, |\vec{u}_k| \, \|^{m+1}_{\frac{(m+1)n}{n-2}} \leq \sum_{j=1}^{N} C(n,m) [\| u_k^j \|_{\frac{(m+1)n}{n-2}}]^{m+1}$$

$$\leq \sum_{j=1}^{N} \lambda \int_{\mathbb{R}^n} |f^i(x, \vec{u})| \cdot [u^j]^m dx$$

$$\leq K \int_{\mathbb{R}^n} C(x) |\vec{u}|^{m+\gamma}$$

and the proof then follows the arguments of the scalar case, [AY] applied to $|\vec{u}|$. Alternatively we may use the results of Egnell, [Eg].

1.4. Positive Solutions for Small λ

We consider, in this section, the problem of establishing the existence, for small $\lambda > 0$, of positive solutions for (1.1.1) without several of the assumptions made in Section 1.3. In particular we no longer require quasi-irreducibility

and our systems may now also allow the existence of nonnegative nontrivial solutions. Our idea is to use the product properties of the Leray-Schauder Degree in order to obtain results for systems which are perturbations of products of simpler systems which satisfy the conditions of Section 1.3. Specifically, we consider the case where such elementary subsystems are simply scalar equations, but we could employ for example $N \times N$ irreducible systems as our starting situations, at the expense of technical complications in the presentation.

Our assumption is as follows:

(II-1) There exist nonnegative functions $P^j(x)$, $q^j(x)$ and $\eta^j(x)$, $j = 1, \ldots, N$ such that $P^j(\xi^j)^{\gamma_j}$ satisfies the monotonicity conditions of Section 1.3, $0 < P^j \in L^p \cap L^\infty$ for some $p < \frac{n}{2}$, $q^j \in L^{\frac{2n}{(n+2)-\gamma_j(n-2)}} \cap L^\infty$, $\eta^j \in L^{\frac{2n}{(n+2)-\theta_j(n-2)}}$ with $1 < \theta_j < \gamma_j$ and for all $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ such that

$$|f^j(x, \vec{\xi}) - P^j(x)(\xi^j)^{\gamma_j}| \leq q^j(x)\varepsilon|\vec{\xi}|^{\gamma_j} + \eta^j(x)K(\varepsilon)|\vec{\xi}|^{\theta_j} \tag{1.4.1}$$

for any $\vec{\xi} \geq 0$.

As before, we assume $1 < \gamma_j < \frac{n+2}{n-2}$.

LEMMA 1.10. *(a) For each $j = 1, \ldots, N$ there exists an annulus $A_j \subset E$ such that*

$$deg\left(u^j - ((-\Delta)^{-1}[P^j[(u^j)^+]^{\gamma_j}], A_j, 0\right) \neq 0.$$

(b) Let

$$S(\vec{u}) = (-\Delta)^{-1} \begin{pmatrix} P^1[(u^1)^+]^{\gamma_1} & & 0 \\ & \ddots & \\ 0 & & P^N[(u^N)^+]^{\gamma_N} \end{pmatrix}$$

and

$$A = A_1 \times \cdots \times A_N.$$

Then $deg\,(\vec{u} - S(\vec{u}), A, \vec{0}) \neq 0.$

PROOF: Part (a) immediately follows from the assumptions on P^j and the results of Section 1.3. Part (b) is essentially an application of the product property of the Leray-Schauder Degree [Ze, Vol. 1, p. 573].

Our existence theorem in this section is proved for the system:

$$(1.1.1)' \qquad -\Delta u^i = \lambda_i f^i(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) > \vec{0}$. We observe however that if $\gamma_1 = \gamma_2 = \cdots = \gamma_N = \gamma$ in (II-1) then $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \lambda$.

THEOREM 1.11. *If (II-1) holds then* $(1.1.1)'$ *has a positive solution for* $\vec{\lambda} > \vec{0}$, *small.*

PROOF: Let T denote the set of solutions to $\vec{u} = S(\vec{u})$ in A. We observe immediately that T is a compact set in E. Let $M \subset A$ denote a neighborhood of T chosen as follows: Let $J(u^j) = \int_B [(u^j)^+]^{\frac{2n}{n-2}}$ for

$j = 1, \ldots, N$ and some fixed ball $B \subset \mathbb{R}^n$. Since T is compact we observe that for each j, $J(u^j)$ assumes its positive infimum on T. We choose M small enough so that if $\vec{u} \in M$ then $J(u^j) > \alpha_1 > 0$ for some constant α_1 and any $j = 1, \ldots, N$. Now $\deg\left(\vec{u} - S(\vec{u}), M, \vec{0}\right) \neq 0$ and we observe that (1.4.1) may be rewritten as

$$(1.4.2) \qquad |\alpha^{\gamma_j} f^j(x, \tfrac{\vec{\xi}}{\alpha}) - P^j(x)(\xi^j)^{\gamma_j}| \leq \varepsilon q^j(x)|\vec{\xi}|^{\gamma_j} + K(\varepsilon)\alpha^{\gamma_j - \theta_j}\eta^j(x)|\vec{\xi}|^{\theta_j}$$

for any $\alpha > 0$.

Let $P(\vec{u}) = (-\Delta)^{-1}\{[\text{Diag}\,(\alpha^{\gamma_j}) \cdot \vec{f}(x, \frac{\vec{u}^+}{\alpha})] - \text{Diag}\,\left(P^j(x)[(u^j)^+]^{\gamma_j}\right)\}$. We observe that (1.4.2) and (II-1) imply that $P : E \to E$ is continuous, compact and by choosing first $\varepsilon > 0$ small enough and then α small, we conclude:

$$\deg\left(\vec{u} - S(\vec{u}) - P(\vec{u}), M, \vec{0}\right) \neq 0,$$

i.e., there exists a solution $\vec{u}$ to

$$-\Delta\vec{u} = \text{Diag}\,(\alpha^{\gamma_j}) \cdot \vec{f}(x, \frac{\vec{u}^+}{\alpha}).$$

Observe that $\vec{u} \geq \vec{0}$ and $\vec{u} \not\equiv \vec{0}$. Indeed, since $\vec{u} \in M$ then $J(u^i) > 0$ and hence $(u^i)^+ \not\equiv 0$, and consequently $u^i > 0$ by the (scalar) Maximum Principle. Finally, setting $\vec{v} = \frac{\vec{u}}{\alpha}$ gives the desired result. Note that since $\vec{u} \in E$, we have that $\vec{u} \in L^\infty$ and $\vec{u} \to \vec{0}$ at ∞ by Lemma 1.9.

1.5. Examples

We conclude this chapter, in this section, with some illustrative examples.

EXAMPLE 1: Consider the system

$$(1.5.1) \qquad \begin{cases} -\Delta u &= \lambda P^1(x)(u+v)^\sigma \\ -\Delta v &= \lambda P^2(x)(u+v)^\sigma \end{cases} \quad \text{in} \quad \mathbb{R}^n,$$

$\lambda > 0$. Let $1 < \sigma < \frac{n+2}{n-2}$. If $0 < P^i(x)$ is such that there exists a number $\gamma_0 > 0$ such that

$$(1.5.2) \qquad \begin{cases} \text{if} & x_j < -\gamma_0 \quad \text{and} \quad x_j < x_j' \quad \text{with} \quad x_j + x_j' \le 0, \quad \text{then} \\ & P^i(x_1, \dots, x_j, \dots, x_n) \le P^i(x_1, \dots, x_j', \dots, x_n) \\ \text{and} & \\ \text{if} & x_j > \gamma_0 \quad \text{and} \quad x_j > x_j' \quad \text{with} \quad x_j + x_j' \ge 0, \quad \text{then} \\ & P^i(x_1, \dots, x_j, \dots, x_n) \le P^i(x_1, \dots, x_j', \dots, x_n), \end{cases}$$

$j = 1, \dots, n$; $P^i \in L^q \cap L^\infty$, $q < \frac{n}{2}$ and $P^i(x) = O(|x|^{-\beta})$ at ∞ with $\beta + \sigma(n-2) > (n+1)$, $i = 1, 2$; then for any $\gamma > \gamma_0$, say, with $x_j > \gamma$, we have $(x^{\gamma_j})_j < x_j$ and $x_j + (x^{\gamma_j})_j > 0$ where x^{γ_j} denotes the reflection of x about the plane $x_j = \gamma$. Thus by taking $x_j' = (x^{\gamma_j})_j$ we observe that all the conditions (I-1) -(I-4) are satisfied and system (1.5.1) has a positive decaying solution (u, v) by Corollary 1.8 and Lemma 1.9, for all $\lambda > 0$.

EXAMPLE 2: We consider again system (1.5.1) with all the above conditions satisfied except for (1.5.2) which we replace with:

$$(1.5.3) \qquad \begin{cases} P^i(x) = \sum_{k=1}^{n} P_k^i(|x_k|), \quad P_k^i \text{ is even} \\ P_k^i \in C^1 \quad \text{with} \quad (P_k^i)' \le 0, \quad \text{near} \quad \infty, \quad i = 1, 2. \end{cases}$$

Under the above conditions, the conclusion of Example 1 holds.

EXAMPLE 3: To illustrate the perturbation arguments, let $P^1(x)$, $P_2(x)$ be as above. We have existence for

$$(1.5.4) \qquad \begin{cases} -\Delta u &= P^1(x)u^{\gamma} + \varepsilon_1 q^1(x)u^{\gamma_1}v^{\gamma_2} \\ -\Delta v &= P^2(x)v^{\gamma'} + \varepsilon_2 q^2(x)u^{\gamma_1'}v^{\gamma_2'} \end{cases}$$

if $\varepsilon_1, \varepsilon_2$ are small enough and $1 < \gamma_1 + \gamma_2 < \gamma < \frac{n+2}{n-2}$, $1 < \gamma_1' + \gamma_2' < \gamma' < \frac{n+2}{n-2}$ with $0 \leq q^1 \in L^{\frac{2n}{(n+2)-\gamma(n-2)}} \cap L^{\infty}$ and $0 \leq q^2 \in L^{\frac{2n}{(n+2)-\gamma'(n-2)}} \cap L^{\infty}$. Setting $(u, v) = \alpha(\widetilde{u}, \widetilde{v})$ then gives

$$(1.5.5) \qquad \begin{cases} -\Delta \widetilde{u} &= \lambda_1[P^1(x)\widetilde{u}^{\gamma} + q^1(x)\widetilde{u}^{\gamma_1}\widetilde{v}^{\gamma_2}] \\ -\Delta \widetilde{v} &= \lambda_2[P^2(x)\widetilde{v}^{\gamma'} + q^2(x)\widetilde{u}^{\gamma_1'}\widetilde{v}^{\gamma_2'}] \end{cases}$$

if we choose $\varepsilon_1 = \alpha^{\gamma-(\gamma_1+\gamma_2)}$, $\lambda_1 = \alpha^{\gamma-1}$, $\varepsilon_2 = \alpha^{\gamma'-(\gamma_1'+\gamma_2')}$ and $\lambda_2 = \alpha^{\gamma'-1}$. So by choosing α sufficiently small, we have that (1.5.5) has a positive (componentwise) solution $(\widetilde{u}, \widetilde{v})$ for $\lambda_1, \lambda_2 > 0$, small. Note that system (1.5.5) also has nontrivial nonpositive solutions, for example $(\widetilde{u}, 0)$ where $\widetilde{u} > 0$ solves

$$-\Delta \widetilde{u} = \lambda_1 P^1(x)\widetilde{u}^{\gamma}.$$

The existence of a solution for such a scalar case also follows from Mountain Pass arguments.

CHAPTER 2

POSITIVE SOLUTIONS WITH EXPONENTIAL DECAY

2.1. Introduction

This chapter deals with the extension of the results obtained in Chapter 1 to the case in which the operator $-\Delta$ in (1.1.1) is replaced with $-\Delta + 1$. Thus, we consider the question of existence of positive solutions with exponential decay for semilinear elliptic systems of the type:

$$\ell u^i = -\Delta u^i + u^i = \lambda f^i(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n \tag{2.1.1}$$

where $1 \leq i \leq N$, $n > 2$, $\lambda > 0$ and x, $\vec{u}$ and Δ are as in Chapter 1. Also as in Chapter 1, we assume that $f^i(x, \vec{u})$ is superlinear and subcritical at ∞, for each i; and observe that (2.1.1) does not have variational structure. We give further conditions on f^i below.

The discussions in this chapter, apart from being a natural progression from those in Chapter 1, are independently motivated by the results of Chaljub-Simon and Volkmann [CV] for scalar equations – via our observation that the assumptions, made by the authors, on the nonlinear terms do not appear to be sufficient to guarantee the existence of their results. We shall thus improve upon these results, in this chapter, by making an additional assumption which we believe is necessary for the results obtained by the authors to hold. We shall also use weighted L^p spaces as against exponential weighted spaces of classically differentiable functions used by the authors. Existence questions for

systems of the type (2.1.1) have been considered by Noussair and Swanson [NS3] under the assumptions that the system has variational structure.

Our aim in this chapter is to obtain existence results for positive solutions with exponential decay for system (2.1.1) under the assumption that the system is nonvariational. Our method is basically identical to that used in Chapter 1 – it involves the use of monotonicity results, global a priori estimates and topological degree arguments. The addition of the linear term does however give stronger results with fewer assumptions.

The layout of this chapter is as follows: in Section 2.2, we introduce the spaces that we shall work with, and state some useful embedding theorems. We present our results in Section 2.3, and conclude the chapter with examples which compare our results with some earlier ones in Section 2.4.

2.2. Background Materials

In this section, we introduce the relevant Banach Spaces which we will need in this chapter. We will also state some useful embedding results.

E, in this chapter, will denote the Sobolev space obtained as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{1,2}^2 = \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2)dx. \tag{2.2.1}$$

We recall the following definitions from the paper by M.S. Berger and M. Schechter [BS]. Let $\omega(x)$ be a measurable function on $\mathbb{R}^n$. For $1 \leq p < \infty$ and

$0 < \alpha < n$, we set

$$M_{\alpha,p}(\omega) = \sup_{x \in \mathbb{R}^n} \int_{|y|<1} |\omega(x-y)|^p |y|^{\alpha-n} dy.$$

$M_{\alpha,p}$ will denote the set of functions ω which satisfy $M_{\alpha,p}(\omega) < \infty$. We now recall the following embedding results of Berger and Schechter.

PROPOSITION 2.1. *Assume that* $2 \leq q < \frac{2n}{n-2}$. *Let* $\alpha > 0$ *be such that* $\alpha < n - \frac{q}{2}(n-2)$. *Then there is a constant* C *depending only on* q *and* n *such that*

$$\|\omega u\|_{L^q} \leq C M_{\alpha,q}(\omega)^{1/q} \|u\|_{1,2} \tag{2.2.2}$$

$u \in E$, $\omega \in M_{\alpha,q}$.

PROPOSITION 2.2. *Let* q, n *and* α *satisfy the hypotheses of Proposition 2.1. Suppose* $\omega(x)$ *is a function in* $M_{\alpha,q}$ *such that*

$$\int_{|y|<1} |\omega(x-y)|^q dy \to 0 \quad \text{as} \quad |x| \to \infty.$$

Then multiplication by ω *is a compact operator from* E *to* L^q. *Thus if* $\{u_k\}$ *is a sequence of functions in* E *with uniformly bounded norms, then* $\{\omega u_k\}$ *has a subsequence converging in* $L^q(\mathbb{R}^n)$.

The proofs of Propositions 2.1 and 2.2 are found in [BS].

Now, let Ω be any domain in $\mathbb{R}^n$. We say that a function $u(x)$ on Ω is in E_Ω if it is the restriction to Ω of a function in E. We set $M_{\alpha,p}(\omega,\Omega) = M_{\alpha,p}(\omega|_\Omega)$ with

$$\omega|_\Omega(x) = \begin{cases} \omega(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Let $M_{\alpha,p}(\Omega)$ denote the set of those functions ω on Ω such that $M_{\alpha,p}(\omega,\Omega) < \infty$. We have:

PROPOSITION 2.3. *Let Ω be any domain in $\mathbb{R}^n$. Under the hypotheses of Proposition 2.1,*

$$\|\omega u\|_{L^q(\Omega)} \le C M_{\alpha,q}(\omega,\Omega)^{1/q} \|u\|_{1,2,\Omega}, \tag{2.2.3}$$

$u \in E_\Omega$, $\omega \in M_{\alpha,q}(\Omega)$ and C the constant in (2.2.2).

PROPOSITION 2.4. *Under the hypotheses of Proposition 2.3, let ω be a function in $M_{\alpha,p}(\Omega)$ such that*

$$\int_{|x-y|<1;y\in\Omega} |\omega(y)|^q dy \to 0 \quad \text{as} \quad |x| \to \infty.$$

Then multiplication by ω is a compact operator from E_Ω to $L^q(\Omega)$.

Again, the proofs are in [BS].

As in Chapter 1, the various norms of the vector function $\vec{u}$ are defined in terms of the norms of the components. Notations introduced in Chapter 1 continue to hold here.

2.3. Results

We present our results in this section. We show, in particular, that system (2.1.1) admits positive solutions with exponential decay for any $\lambda > 0$. The functions f^i $(1 \leq i \leq N)$ continue to satisfy all the assumptions stated in Chapter 1 except for the following changes to the conditions on the function $C(x)$.

(A3) $C(x) \in L^\infty$ with

$$M_{\alpha,1}(C, \mathbb{R}^n \backslash B_R) \to 0 \quad \text{as} \quad R \to \infty$$

for some $0 < \alpha < n - \frac{(\gamma+1)}{2}(n-2)$. Here B_R is the ball in $\mathbb{R}^n$ with center 0 and radius R.

REMARK: The monotonicity assumption of the type (I-1) was not made by Chaljub-Simon and Volkmann [CV]. This condition was used in a crucial way in the proof of Theorem 1.3 via Lemma 1.4.

We observe that the arguments of the proofs of results in this chapter are basically the same as in Chapter 1 with $\|C\|_{L^s(\mathbb{R}^n \backslash B_R)}$ replaced by $M_{\alpha,1}(C, \mathbb{R}^n \backslash B_R)$. To this end, we conclude, by using the exponential decay result of Egnell [Eg, Theorem 5] (see also [GNN2]), that Theorem 1.3 and Proposition 1.5 hold for positive solutions $\vec{u}$ of (2.1.1). Finally, we mention that Proposition 1.2, which was used in the proof of Theorem 1.3 in Chapter 1, is replaced by the following lemma in this chapter.

LEMMA 2.5. *Let* $u^i > 0 \quad (1 \le i \le N)$ *satisfy (2.1.1). Suppose* $\{x^k\} \in \mathbb{R}^n$ *is a sequence tending to infinity in* $\mathbb{R}^n$ *and* $\xi = \lim_k \frac{x^k}{|x^k|}$, $\xi \equiv (\xi_1, \xi') = (\xi_1, \xi_2, \ldots, \xi_n)$. *Then*

(a) $\lim_{k\to\infty} |x^k|^{\frac{(n-1)}{2}} e^{|x^k|} u^i(x^k) = \sqrt{\pi/2} \int dy\, g^i(y) e^{\xi\cdot y}$.

(b) Suppose $\xi_1 = 0$ *and that* μ_k^i *is a sequence of real numbers converging to a number* μ^i *and* $\mu_k^i > x_1^k$. *Let* z^k *be the reflection of* x^k *in the plane* $x_1^k = \mu_k^i$, *i.e.,* $z^k = (2\mu_k^i - x_1^k, x_2^k, \ldots, x_n^k)$. *Then*

$$\lim_{k\to\infty} \frac{|x^k|}{\mu_k^i - x_1^k} |x^k|^{\frac{(n-1)}{2}} e^{|x^k|}\left(u^i(x^k) - u^i(z^k)\right) = \sqrt{2\pi} \int dy\, e^{\xi'\cdot y'} g^i(y)(\mu^i - y_1).$$

(c) If $x_1^k \to +\infty$, *then*

$$\frac{|x^k|^{\frac{(n+1)}{2}}}{x_1^k} e^{|x^k|} u^i_{x_1}(x^k) \to -\sqrt{\pi/2} \int dy\, e^{\xi\cdot y} g^i(y).$$

PROOF: The same as for Proposition 4.2 in [GNN2].

Let us observe that since $M_{\alpha,1}(C, \mathbb{R}^n) < \infty$ and $0 < \alpha < 2$, the relation $\|\vec{u}\|_E \sim \| \, |\vec{u}| \, \|_{L^\infty}$ (in the sense that a bound on one norm implies a bound on the other) hold for positive solutions $\vec{u}$ of (2.1.1) by arguing as in Chapter 1. We now state our existence result.

THEOREM 2.6. *Assume that* $\vec{f}(x,\vec{u})$ *satisfies the assumptions stated in Chapter 1 and that* $C(x)$ *in (I-3) satisfies (A3). Then (2.1.1) has a positive solution* $\vec{u}$ *decaying exponentially at* ∞.

PROOF: In view of all the observations made above, the proof follows as in the case of Theorem 1.7 by degree arguments – Proposition E. We however observe that, here, the compactness and continuity arguments follow from the use of the embedding results stated above. In particular, we note that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_R} C|\vec{u}|^{\gamma+1} &= \left(\|C^{1/\gamma+1}\cdot\vec{u}\|_{L^{\gamma+1}(\mathbb{R}^n\setminus B_R)}\right)^{\gamma+1}\\ &\le KM_{\alpha,\gamma+1}\left(C^{\frac{1}{(\gamma+1)}},\mathbb{R}^n\setminus B_R\right)\|\vec{u}\|_{1,2}^{\gamma+1}\\ &= KM_{\alpha,1}(C,\mathbb{R}^n\setminus B_R)\|\vec{u}\|_{1,2}^{\gamma+1}.\end{aligned}$$

By (A3), $M_{\alpha,1}(C,\mathbb{R}^n\setminus B_R)$ can be made arbitrarily small for R sufficiently large.

2.4. Examples

We conclude the chapter by giving explicit examples to illustrate our results. Some of our examples compare our results to earlier ones.

EXAMPLE 1: Consider first the scalar equation

$$-\Delta u + u = b(x)u^{\gamma}, \quad x\in\mathbb{R}^n. \tag{2.4.1}$$

Suppose $1<\gamma<\frac{n+2}{n-2}$, and $b(x)$ satisfies the conditions

(i) $b \in L^\infty(\mathbb{R}^n)$ and $\|b\|_{L^1(B_1(x))} \to 0$ as $|x| \to \infty$ and

(ii) there is a number $\widetilde{\mu} > 0$ such that if $x_j > \widetilde{\mu}$ and x_j' is any number such that $x_j' < x_j$ and $x_j' + x_j \geq 0$, then

$$b(x_1, x_2, \ldots, x_j, \ldots, x_n) \leq b(x_1, x_2, \ldots, x_j', \ldots, x_n);$$

if $x_j < -\widetilde{\mu}$ and $x_j < x_j'$ with $x_j + x_j' \leq 0$ then

$$b(x_1, x_2, \ldots, x_j, \ldots, x_n) \leq b(x_1, \ldots, x_j', \ldots, x_n).$$

Then, since by $0 < \alpha < n - \frac{(\gamma+1)}{2}(n-2)$, $b \in L^\infty$ and Hölder's Inequality, we have

$$\int_{|y-x|\leq 1} |b(y)|\,|y-x|^{-(n-\alpha)} dy \leq C(\|b\|_\infty, \alpha)[\|b\|_{L^1(B_1(x))}]^\beta \to 0$$

as $|x| \to \infty$, for $0 < \beta < \frac{\alpha}{n}$; we conclude that $b(x)$ satisfies condition (A3). Note, as in Example 1 in Chapter 1, that (ii) implies b satisfies (I-1) – the monotonicity conditions in x. Therefore, by Theorem 2.6, equation (2.4.1) has a positive solution u decaying exponentially at ∞. Chaljub-Simon and Volkmann [CV] considered equations of the type (2.4.1) and required exponential decay of $b(x)$.

REMARK: As mentioned above, Chaljub-Simon and Volkmann did not make an assumption of the type (ii) above, which makes the result of the type obtained in Theorem 1.3 inapplicable in their situation. We note that the number $\widetilde{\mu}$

(appearing in (ii)) and used in Theorem 1.3 is independent of the solutions and so without such an assumption, we conjecture that there may not be a global uniform bound for all positive solutions of (2.4.1) in the entire space $\mathbb{R}^n$.

EXAMPLE 2: We now consider the system

$$(2.4.2) \qquad \begin{cases} -\Delta u + u = b^1(x)(u+v)^\gamma \\ -\Delta v + v = b^2(x)(u+v)^\gamma \end{cases} \quad \text{in} \quad \mathbb{R}^n$$

where $1 < \gamma < \frac{n+2}{n-2}$. If $0 < b^1,\ b^2$ satisfy the same conditions as b above, then (2.4.2) has a positive solution (u, v) decaying exponentially at ∞. Observe that system (2.4.2) is quasi-irreducible.

EXAMPLE 3: Consider again system (2.4.2). Let $0 < b^1,\ b^2$ satisfy condition (i), as $b(x)$ in Example 1. In place of condition (ii), let $b^1,\ b^2$ satisfy instead the following: $b^1,\ b^2$ are radial near infinity and $(b^1)',\ (b^2)' \leq 0$. Then the same conclusion as in Example 2 holds.

CHAPTER 3

SEMILINEAR ELLIPTIC EQUATIONS WITH THE OPERATORS ADMITTING FIRST-ORDER TERMS

3.1. Introduction

The results we obtained in Chapters 1 and 2 depended, in a crucial way, on the nature of the operators considered – in particular, operators did not contain first-order terms. Our desire in this chapter is to extend our results to the case of second order elliptic operators admitting first-order terms and without the above postulated monotonicity condition on $\vec{f}$ at $|x| = \infty$. It seems to us that only very little is available in this case – even for scalar equations, with the nonlinearities assumed to be superlinear and subcritical at ∞. Therefore we shall start with the investigation of existence results for scalar equations in Section 3.3 and give extensions to systems in Section 3.4 of this chapter. To this end, we consider the existence of positive decaying solutions of semilinear elliptic equations of the type:

$$\ell u = f(x,u) \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3, \tag{3.1.1}$$

where ℓ denotes the elliptic operator

$$\ell u = -\Delta u + \sum_{j=1}^{n} b_j(x) D_j u \tag{3.1.2}$$

with $b_j \in C^1 \cap L^\infty(\mathbb{R}^n)$. We shall consider also the system

$$\ell^\nu u^\nu = f^\nu(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n, \quad 1 \le \nu \le N \tag{3.1.3}$$

where $x = (x_1, \dots, x_n)$, $n \ge 3$, $\vec{u} = (u^1, u^2, \dots, u^N)$ and ℓ^ν stands for the elliptic expression

$$\ell^\nu u^\nu = -\Delta u^\nu + \sum_{j=1}^{n} b_j^\nu(x) D_j u^\nu, \tag{3.1.4}$$

$b_j^\nu \in C^1 \cap L^\infty(\mathbb{R}^n)$. As in the previous chapters, the functions $f(x,u)$, $f^\nu(x,\vec{u})$, $i \le \nu \le N$ are assumed to be superlinear and subcritical at ∞. More precise conditions on the coefficient functions b_j, b_j^ν and the nonlinearities f, f^ν will be given below.

The question of existence and nonexistence of positive decaying solutions for the second order semilinear elliptic problems:

$$\begin{cases} \widetilde{\ell} u = -\sum_{i,j=1}^n D_i(a_{ij}(x) D_j u) + \sum_{j=1}^n b_j(x) D_j u + C(x) u = f(x,u) \\ u > 0, \quad \lim\limits_{|x| \to \infty} u = 0 \end{cases}$$

in $\mathbb{R}^n$, has been investigated by several authors in the case in which $\widetilde{\ell}$ has a variational form, that is, the coefficients b_j are all zero. We mention in particular the papers of: Atkinson and Peletier [AP], Berger [Be], Berestycki and Lions [BL], Coffman and Marcus [CM], Ding and Ni [DN1] [DN2], Fukagai [Fu], Furusho [Fur 2], Kawano [Ka], Kusano and Naito [KN], Kusano and Oharu

[KO], Ni [Ni1] [Ni2] and Noussair and Swanson [NS6][NS7]. The prototype problem considered by these authors is

$$-\Delta u + a(x)u = f(x,u), \tag{3.1.5}$$

with $f(x,u) = b(x)u^{\alpha}, 1 < \alpha < \frac{n+2}{n-2}$, and $a(x) \equiv \widetilde{a}$ (a positive constant) for most of the considerations. Results were often obtained either for the radially symmetric case, $f(x,u) = f(|x|,u)$, or for the case in which $f(x,u)$ admitted suitable radial majorants. The methods employed included bounded subdomains approximation of $\mathbb{R}^n$, ODE theory and variational methods — with either the use of Sobolev spaces of highly symmetric functions which admit compact embeddings or rearrangement of functions to construct compact minimizing sequences. More details on earlier references will be given in Chapter 4.

It seems only very few results are available in the case in which the elliptic operators under consideration admit first-order terms. In the papers of Allegretto [All], and Furusho [Fur1] [Fur2] where such operators were considered, the nonlinearities were either assumed to be suitable bounded or to be sublinear, or solutions did not tend to zero.

In this chapter, we wish to establish conditions under which (3.1.1) and (3.1.3) have positive decaying solutions – solutions that tend to zero. We will not require our nonlinearities to be radially symmetric, to admit radial majorant or to satisfy conditions for variational structure in the systems case. Our main tools are weighted spaces, a priori estimates and topological degree arguments.

Unlike the situations in Chapters 1 and 2, we remark that our procedure in this chapter only enables us to achieve a growth of order $|\vec{u}|^\beta$, $1 < \beta < \frac{n}{n-2}$. We recall however, there is the advantage that we no longer require the nonlinearities to satisfy monotonicity conditions.

This chapter is organized as follows: we set up our weighted spaces and give some preliminary results in Section 3.2. Our first existence result will be presented in Section 3.3. We prove, in particular, that (3.1.1) admits a positive decaying solution if $|f(x,t)| \leq P(x)|t|^s$ for $1 < s < \frac{n}{n-2}$ with $P \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ suitably chosen. In Section 3.4, we extend the results obtained in Section 3.3 for scalar equations to systems of the type (3.1.3). We conclude the chapter, in Section 3.5, with examples.

3.2. Notations and Preliminary Results

In this chapter, we shall take the space E to be the same as in Chapter 1, i.e., the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^n} (|\nabla u|^2 + \sigma u^2) dx \tag{3.2.1}$$

where $\sigma(x) = (1+|x|^2)^{-1}$. If ρ is a positive (Lebesgue) measurable function on $\mathbb{R}^n$, we denote by $L^2(\mathbb{R}^n, \rho)$ the weighted L^2 space with weight ρ. $W^{1,2}(\mathbb{R}^n)$ will denote the standard Sobolev space. We recall some of the properties of E given in Lemma 1.0 – restated as 3.1 for convenience. Others are given below and further can be found in ([AY], [Al2], [BF2], [Fo2]).

LEMMA 3.1.

(a) Sobolev's Inequality: $\|u\|_{\frac{2n}{n-2}} \leq C\|\nabla u\|_2$ *holds in* E

(b) If $u \in E$ *then* $u^+, u^- \in E$.

LEMMA 3.2. *Let* γ *be a positive, measurable function on* $\mathbb{R}^n$, *satisfying the following assumptions:*

(i) $\sigma \cdot \gamma \in L^r_{\text{loc}}(\mathbb{R}^n)$ *with* $r > \frac{n}{2}$,

(ii) $\lim_{|x| \to +\infty} \gamma(x) = 0.$

Then the embedding $E \hookrightarrow L^2(\mathbb{R}^n, \sigma \cdot \gamma)$ is compact.

PROOF: Essentially follows the proof of Theorem 1.2 of [BF2].

By recalling Hardy's inequality, which was stated in Chapter 1, we observe that $E \sim \widetilde{E}$, where $\widetilde{E}$ is the Hilbert space completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the inner product $[\cdot, \cdot]$ given by

$$[u, v] = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx, \quad u, v \in C_0^\infty(\mathbb{R}^n). \tag{3.2.2}$$

As in the previous chapters, the norms of the vector functions $\vec{u}$ are defined in terms of the norms of the components. Thus, the norm of $\vec{u} \in E$ is given by

$$\|\vec{u}\|_E^2 = \sum_{i=1}^{N} \|u^i\|_E^2. \tag{3.2.3}$$

As always, vector inequalities will be understood to hold componentwise.

3.3 Existence Result for Equation (3.1.1)

We present our existence result for (3.1.1) in this section. First we state our hypotheses on f and the coefficient functions b_j appearing in (3.1.2) and obtain certain a priori estimates.

(A1) $f \in C^{\alpha}_{loc}(\mathbb{R}^n \times \mathbb{R})$ and $0 \le f(x,t)$ in $\mathbb{R}^n \times \mathbb{R}_+$, $f(x,0) \equiv 0$;

(A2) $\underline{\lim}_{t \to +\infty} \frac{f(x,t)}{tP(x)} > \lambda_1$ uniformly for $x \in \mathbb{R}^n$, with λ_1 specified below;

(A3) $f(x,t) \le P(x)|t|^s$ with $1 < s < \frac{n}{n-2}$;

(A4) There exists an open ball B centered at the origin and a constant K_1 such that for $x \in \mathbb{R}^n \backslash B$,

$$|\nabla b_j|^2 + |b_j(x)|^2 \le K_1 \sigma^{2+\epsilon}(x) \quad (j = 1, \ldots, n), \quad \epsilon > 0;$$

(A5) $0 \le P(x) \in L^{q_0}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$, $q_0 = \frac{2n}{n+2}$, is such that there exists a non-negative measurable function $\gamma(x)$ on $\mathbb{R}^n$ with $\lim_{|x| \to \infty} \gamma(x) = 0$, $\gamma(x) \in L^{\frac{2}{1-\theta}}(\mathbb{R}^n)$, $\theta = \frac{s(n-2)}{n}$; and $P(x) = \sigma(x) \cdot \gamma(x)$, $\sigma(x)$ as specified above; we observe that since $\gamma \in L^{\infty}$, it actually suffices to assume that $\gamma \in L^{\xi}$ for some $\xi > 2$; since a simple calculation then shows that $\gamma \in L^{\frac{2}{1-\theta}}$.

(A6) $(\ell u, u) \ge \epsilon' \|u\|_E^2$ for some constant ϵ'.

We remark that λ_1 in (A2) is the principal eigenvalue of the problem

(3.3.1) $$\ell^* v = \lambda P(x) v \quad \text{in} \quad \mathbb{R}^n$$

where ℓ^* is the formal adjoint operator to ℓ. The corresponding eigenfunction J to the eigenvalue λ_1 is positive, is contained in E and satisfies $\lim_{|x|\to\infty} J(x) = 0$; see for instance [Al2]. However, that J, λ_1 exist is a consequence of the compactness and positivity on E of the operator T (this we shall prove below) given by $Tu = \ell^{-1}[P(x)u]$ – via the Krein-Rutman theorem.

LEMMA 3.3. *Under assumptions (A4) and (A6), the operator T on E given by*

$$Tu = \ell^{-1}[P(x)u]$$

is compact and positive.

PROOF: That ℓ has a continuous inverse ℓ^{-1} is a consequence of the continuity on $E \times E$ of the associated bilinear form (see [BF2]) and the assumption (A6) which guarantees that the operator ℓ is one-to-one on E. First, we show the compactness of T by assuming $P(x) > 0$.

Let us consider

$$g : u \in L^2(\mathbb{R}^n, P) \mapsto g(u) \in E'(\sim E)$$

where

$$\langle g(u), v\rangle = \int_{\mathbb{R}^n} P(x)u(x)v(x)dx, \quad \text{for all} \quad v \in E.$$

g is continuous from $L^2(\mathbb{R}^n, P)$ into E'. In fact for all $u \in L^2(\mathbb{R}^n, P)$, $v \in E$:

$$|\langle g(u), v\rangle| \leq \|u\|_{L^2(\mathbb{R}^n,P)}\|v\|_{L^2(\mathbb{R}^n,P)}$$

$$\leq \|u\|_{L^2(\mathbb{R}^n,P)}\|v\|_E.$$

By Lemma 3.2, the embedding $i : E \to L^2(\mathbb{R}^n, P)$ is compact. Thus by observing that

$$T = \ell^{-1} \circ g \circ i,$$

we conclude that T is compact. For the general case of P, we set $P_\varepsilon = P + \varepsilon\varphi$, $0 < \varphi \in C^\infty$, argue with P_ε as above and let $\varepsilon \to 0$. By arguing as in [AY], we have that $\ell u \geq (\not\equiv) 0$ implies $u > 0$ for $u \in E$, whence T is positive since $P(x) \geq 0$.

Next, we obtain an a priori estimate.

PROPOSITION 3.4. *Assume that hypotheses (A1) - (A6) are satisfied. If* $0 \leq u \in E \cap L^\infty$ *satisfies*

$$\begin{cases} \ell u = f(x,u) + tP(x)J, \\ u \to 0 \quad \text{as} \quad |x| \to +\infty \end{cases} \tag{3.3.2}$$

then

$$\|u\|_E \leq K$$

where K *is independent of* $t \geq 0$.

Before we give a proof of Proposition 3.4, we first prove a pair of lemmas.

LEMMA 3.5. *Assume that* f *satisfies hypothesis (A2). If* $0 \leq u \in E \cap L^\infty$ *satisfies (3.3.2), for a* $t \geq 0$, *then* t, $\int_{\mathbb{R}^n} f(x,u)J$, $\int_{\mathbb{R}^n} uP(x)J$ *are bounded.*

PROOF: We extend the arguments of Brezis and Turner [BT] for bounded domains. By hypothesis (A2) there exits a constant $K_2 > \lambda_1$ and a u_0 such that $f(x,u) \geq K_2 P(x)u$ for $u \geq u_0$. Then $f(x,u) \geq K_2 P(x)u - C_1 P(x)$ for all $u \geq 0$ since $f(x,u) \geq 0$. For every solution of (3.3.2), we have

$$\lambda_1 \int_{\mathbb{R}^n} uP(x)J = \int_{\mathbb{R}^n} u\ell^* J = \int_{\mathbb{R}^n} J\ell u = \int_{\mathbb{R}^n} Jf(x,u) + \int_{\mathbb{R}^n} JtP(x)J.$$

Thus

$$\begin{aligned}\lambda_1 \int_{\mathbb{R}^n} uP(x)J &= \int_{\mathbb{R}^n} Jf(x,u) + \int_{\mathbb{R}^n} JtP(x)J \\ &\geq K_2 \int_{\mathbb{R}^n} JP(x)u - C_1 \int_{\mathbb{R}^n} P(x)J + \int_{\mathbb{R}^n} JtP(x)J \\ &\geq K_2 \int_{\mathbb{R}^n} JP(x)u - C + \int_{\mathbb{R}^n} JtP(x)J.\end{aligned}$$

(We have used the fact that $\int P(x)J < \infty$ – a consequence of the fact that $P \in L^{\frac{2n}{n+2}}$ and $J \in L^{\frac{2n}{n-2}}$ by Lemma 3.1.) That is,

$$\lambda_1 \int_{\mathbb{R}^n} uP(x)J \geq K_2 \int_{\mathbb{R}^n} JP(x)u - C + \int_{\mathbb{R}^n} JtP(x)J,$$

and we see that t, $\int_{\mathbb{R}^n} JP(x)u$ and $\int_{\mathbb{R}^n} Jf(x,u)$ are bounded.

LEMMA 3.6. *Under the assumptions of Proposition 3.4, there is a constant* K_3 *such that*

$$\|u\|_E \le K_3$$

for every nontrivial nonnegative solution of (3.3.2).

PROOF: First we observe that $\ell^* J \ge 0$, hence $J \ge \frac{C}{|x|^{n-2}}$ for at least $|x| > 1$ by arguing as in Lemma H_1' in [GNN2]. Now, if we multiply (3.3.2) by u and integrate, we have

$$\begin{aligned} \int_{\mathbb{R}^n} u\ell u &= \int_{\mathbb{R}^n} uf(x,u) + t\int_{\mathbb{R}^n} uP(x)J \\ &\le \int_{\mathbb{R}^n} uf(x,u) + C_1. \end{aligned} \tag{3.3.3}$$

However

$$\begin{aligned} \int_{\mathbb{R}^n} uf(x,u) &= \int_{\mathbb{R}^n} (f^\alpha)(f^{1-\alpha}J^\alpha \cdot \frac{u}{J^\alpha}) \\ &= \int_{\mathbb{R}^n} (fJ)^\alpha (f^{1-\alpha} \cdot \frac{u}{J^\alpha}) \\ &\le (\int_{\mathbb{R}^n} fJ)^\alpha (\int_{\mathbb{R}^n} f \frac{u^{\frac{1}{1-\alpha}}}{J^{\frac{\alpha}{1-\alpha}}})^{1-\alpha} \quad \text{for} \quad 0 < \alpha < 1. \end{aligned}$$

By using Lemma 3.5 and Assumption (A5), we have for any large ball B:

$$\int_{\mathbb{R}^n} uf(x,u) \le C_2[(\int_B u^{s+\frac{1}{1-\alpha}} \frac{P(x)}{J^{\frac{\alpha}{1-\alpha}}})^{1-\alpha} + (\int_{\mathbb{R}^n \setminus B} u^{s+\frac{1}{1-\alpha}} \frac{P(x)}{J^{\frac{\alpha}{1-\alpha}}})^{1-\alpha}].$$

By choosing $\alpha = \frac{2}{n}$ and noting that $\frac{P(x)}{J^{\frac{\alpha}{1-\alpha}}} \leq C_3$ in the ball B (with finite volume), we have by setting $\theta = \frac{s(n-2)}{n}$ $(0 < \theta < 1$ since $s < \frac{n}{n-2})$

$$\int_{\mathbb{R}^n} uf(x,u) \leq C_4 \|u\|^{1+\theta}_{\frac{2n}{n-2}} + C_2 \Big(\int_{\mathbb{R}^n \setminus B} u^{s+\frac{1}{1-\alpha}} \frac{P(x)}{J^{\frac{\alpha}{1-\alpha}}} \Big)^{1-\alpha}. \tag{3.3.4}$$

We assume that B is such that $P(x) \not\equiv 0$ in $\mathbb{R}^n \setminus B$ for otherwise we are done. By choosing again $\alpha = \frac{2}{n}$ and using the fact that $J \geq \frac{C}{|x|^{n-2}}$, we have

$$\begin{aligned} \Big(\int_{\mathbb{R}^n \setminus B} u^{s+\frac{1}{1-\alpha}} \frac{P(x)}{J^{\frac{\alpha}{1-\alpha}}} \Big)^{1-\alpha} &\leq C \Big(\int_{\mathbb{R}^n \setminus B} P(x)|x|^2 u^{s+\frac{n}{n-2}} \Big)^{\frac{n-2}{n}} \\ &\leq C \Big(\int_{\mathbb{R}^n \setminus B} P(x)|x|^2 u^{(1+\theta)\frac{n}{n-2}} \Big)^{\frac{n-2}{n}}, \end{aligned}$$

i.e.,

$$\begin{aligned} \Big(\int_{\mathbb{R}^n \setminus B} u^{s+\frac{1}{1-\alpha}} \frac{P(x)}{J^{\frac{\alpha}{1-\alpha}}} \Big)^{1-\alpha} &\leq C \Big(\int_{\mathbb{R}^n \setminus B} (P(x)|x|^2)^{\frac{2}{1-\theta}} \Big)^{(\frac{1-\theta}{2})(\frac{n-2}{n})} \\ &\quad \times \Big(\int_{\mathbb{R}^n \setminus B} u^{\frac{2n}{n-2}} \Big)^{(\frac{n-2}{2n})(1+\theta)} \\ &\leq C_5 \|u\|^{1+\theta}_{\frac{2n}{n-2}} \end{aligned}$$

by hypothesis (A5) since $P(x)|x|^2 \leq \gamma(x) \in L^{\frac{2}{1-\theta}}$.

From (3.3.3), we obtain

$$\int_{\mathbb{R}^n} u\ell u \leq C_7 \|u\|^{1+\theta}_{\frac{2n}{n-2}} + C_1.$$

But, by assumption (A6), we have that

$$\|u\|^2_E \leq C' \int_{\mathbb{R}^n} u\ell u \leq C'_7 \|u\|^{1+\theta}_{\frac{2n}{n-2}} + C'_1.$$

Since $E \hookrightarrow L^{\frac{2n}{n-2}}$ (Lemma 3.1), and $\theta < 1$ we conclude that $\|u\|_E \leq K_3$.

PROOF OF PROPOSITION 3.4: Immediately follows from Lemmas 3.5 and 3.6.

LEMMA 3.7. *If* $0 \leq u \in E$ *satisfies the equation*

$$\ell u = f(x,u) + tP(x)J$$

for some $t \geq 0$, *then* $u \in L^\infty(\mathbb{R}^n)$ *and* $\lim_{|x|\to\infty} u = 0$.

PROOF: Follows similar arguments as in Lemma 1.9 in Chapter 1.

REMARK: We observe that $P \in L^{q_0} \cap L^\infty$, $q_0 = \frac{2n}{n+2}$ implies $P \in L^{\frac{2n}{4+\epsilon(n-2)}}$ for ϵ sufficiently small. Thus, using condition (A6) and arguing as in Chapter 1, we have that $\|u\|_E \sim \|u\|_\infty$.

We now present our existence result. First we recall that ℓ has a continuous inverse ℓ^{-1} – as established in the proof of Lemma 3.3. Let us consider the operator

$$A : E \to E' \quad (\sim E)$$

defined for all $u, v \in E$ by

$$\langle Au, v\rangle = \int_{\mathbb{R}^n} f(x, u(x))v(x). \tag{3.3.5}$$

We have the following lemma:

LEMMA 3.8. *Under the assumptions on the nonlinear term* f, *the operator* A *is compact.*

PROOF: By using the inequality

$$\|Au_k - Au\|_E \leq \{\|f(\cdot, u_k) - f(\cdot.u)\|_{L^{\frac{2n}{n+2}}(\Omega_m)} + \|P\|_{L^{r_0}(\mathbb{R}^n \setminus \Omega_m)}(\|u_k\|_E^s + \|u\|_E^s)]$$

where $\Omega_m = \{x | \ \|x\| < r_m\}$ and $r_0 > \frac{n}{2}$, the result follows by arguing essentially as in [AY].

A solution to our problem will be obtained by applying Krasnosel'skii's type theorem (refer to Proposition E in Section 1.2 of Chapter 1) to the operator equation

$$u = G(v) = \ell^{-1} A(v). \tag{3.3.6}$$

Set $\overline{C} = \{u^+ | u \in E\}$. We observe that $\overline{C} \subset E$ by Lemma 3.1. From above, we have $G : E \to E$ is compact and thus G acting on $\overline{C}$, will have a fixed point V with

$$0 < r \leq \|v\|_E \leq R < \infty$$

provided

(i) $\beta G(v) \neq v, \quad 0 \leq \beta \leq 1$ for $\|v\|_E = r$ and

(ii) $G(v) \neq v - t\widetilde{J}, \quad t \geq 0,$ for $\|v\|_E = R,$ where $\widetilde{J} = \ell^{-1}[P(x)J].$

We now state our existence theorem.

THEOREM 3.9. *Under hypotheses (A1) - (A6), (3.1.1) has at least one nontrivial positive decaying solution.*

PROOF: To prove the theorem, we simply verify conditions (i) and (ii) above. To see that (i) holds for a small r, suppose that $G(u) = \beta u$ with $\beta \geq 1$ and $\|u\|_E = r$. We obtain $\ell u = \beta^{-1} f(x,u)$. Therefore $\int_{\mathbb{R}^n} u\ell^* J = \beta^{-1} \int_{\mathbb{R}^n} f(x,u)J$ and consequently,

$$\lambda_1 \int uP(x)J = \beta^{-1} \int_{\mathbb{R}^n} f(x,u)J \leq \beta^{-1} \int JP(x)u^{s-1}u,$$

since $s > 1$. This implies that

$$u^{s-1}(x_0) \geq \frac{\lambda_1}{\beta} \quad \text{for some} \quad x_0.$$

Thus $\|u\|_E > r$ for some $r > 0$ independent of u. From Lemma 3.5 and Proposition 3.4, we see that t is bounded for any solution of (3.3.2); and $\|u\|_E \leq K$ for any such solution, i.e.; the a priori estimates are uniform in t, from where condition (ii) follows. Finally, the positivity of such a fixed point u, say, follows as in Allegretto and Yu [AY].

3.4. Existence Result for (3.1.3)

In this section, we extend the results obtained in Section 3.3 for scalar equations to systems of the type (3.1.3). As in Section 3.3, the use of Proposition E of Chapter 1 to obtain our existence result, will rely on being able to

obtain a priori estimates for non-negative solutions of

$$(3.4.1) \qquad \begin{cases} \ell^\nu u^\nu = f^\nu(x,\vec{u}) + tP^\nu(x)J^\nu & \text{in } \mathbb{R}^n \\ \lim\limits_{|x|\to\infty} u^\nu = 0, & \nu = 1,\dots,N \end{cases}$$

where $J^\nu(x)$ is chosen to be the eigenfunction corresponding to the principal eigenvalue λ_1^ν of the problem

$$(3.4.2) \qquad \ell^{\nu^*} w^\nu = \lambda^\nu P^\nu(x) w^\nu,$$

and ℓ^{ν^*} is the formal adjoint operator to ℓ^ν.

Before we derive our a priori estimates, we state our assumptions on f^ν and the coefficient functions b_j^ν appearing in (3.1.3) and (3.1.4).

(I) $f^\nu \in C^\alpha_{\ell oc}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $\alpha \in (0,1)$ and $0 \le f^\nu(x,\vec{t})$ in $\mathbb{R}^n \times \mathbb{R}^N_+$, $f^\nu(x,\vec{0}) \equiv 0$, $\nu = 1,\dots,N$.

(II) $\lim\limits_{t^\nu\to\infty} \frac{f^\nu(x,\vec{t})}{t^\nu P^\nu(x)} > \lambda_1^\nu$ uniformly in $t^\mu \ge 0$ for $\nu \ne \mu$, and $x \in \mathbb{R}^n$.

(III) There exists a constant K_1 such that $\sum\limits_{j=1}^n (|\nabla b_j^\nu(x)|^2 + |b_j^\nu(x)|^2) \le K_1\sigma^{2+\varepsilon}(x)$ for each ν.

(IV) $f^\nu(x,\vec{t}) \le P^\nu(x)|\vec{t}|^s$ with $1 < s < \frac{n}{n-2}$.

(V) $0 \le P^\nu(x) \in L^{q_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $q_0 = \frac{2n}{n+2}$, is such that there exists non-negative measurable functions $h^\nu(x)$ on $\mathbb{R}^n$ with $\lim\limits_{|x|\to\infty} h^\nu(x) = 0$, $h^\nu(x) \in L^{\frac{2}{1-\theta}}(\mathbb{R}^n)$, $\theta = \frac{s(n-2)}{n}$ and $P^\nu(x) = \sigma(x)\cdot h^\nu(x)$, $\sigma(x)$ as specified in Section 3.2.

(VI) $\sum_{\nu=1}^{N} (\ell^\nu u^\nu, u^\nu) \geq C' \|\vec{u}\|_E^2$ for some constant C'.

We observe that the existence of the solution J^ν of (3.4.2) follows as in Section 3.3. We note also that $J^\nu > 0$ and $\lim_{|x| \to \infty} J^\nu(x) = 0$ for $\nu = 1, \ldots, N$.

We now start the process of obtaining our a priori estimates with the following lemma:

LEMMA 3.10. *Assume that* f^ν *satisfies hypothesis (II). If* $0 \leq \vec{u} \in E \cap L^\infty$ *satisfies (3.4.1), for a* $t \geq 0$, *then*

$$t, \quad \int_{\mathbb{R}^n} f^\nu(x, \vec{u}) J^\nu, \quad \int_{\mathbb{R}^n} u^\nu P^\nu J^\nu$$

are bounded.

PROOF: The lemma follows precisely as in the case of a single equation, in Section 3.3, by multiplying the ν^{th} equation of (3.4.1) by J^ν and integrating.

Our a priori estimate is embodied in the next proposition:

PROPOSITION 3.11. *Assume that hypothesis (I), (II), (IV) - (VI) are satisfied. If* $0 \leq \vec{u} \in E \cap L^\infty$ *satisfies (3.4.1), then*

$$\|\vec{u}\|_E \leq K$$

where K *is independent of* $t \geq 0$.

PROOF: Following the proof of Lemma 3.6 of Section 3.3, we multiply the ν^{th} equation of (3.4.1) by u^ν and integrate. We obtain by Lemma 3.10

$$\int_{\mathbb{R}^n} u^\nu \ell^\nu u^\nu = \int_{\mathbb{R}^n} u^\nu f^\nu(x,\vec{u}) + \int_{\mathbb{R}^n} tP^\nu J^\nu \leq \int_{\mathbb{R}^n} u^\nu f^\nu + K_2. \tag{3.4.3}$$

By using Hölder's inequality and Lemma 3.10, we estimate the integral on the right side of (3.4.3) as follows:

$$\int_{\mathbb{R}^n} u^\nu f^\nu = \int_{\mathbb{R}^n} (f^\nu)^\alpha \cdot (f^\nu)^{1-\alpha} \frac{(J^\nu)^\alpha \cdot u^\nu}{(J^\nu)^\alpha} \leq (\int_{\mathbb{R}^n} f^\nu J^\nu)^\alpha (\int_{\mathbb{R}^n} f^\nu \frac{(u^\nu)^{\frac{1}{1-\alpha}}}{(J^\nu)^{\frac{\alpha}{1-\alpha}}})^{1-\alpha}$$

for $0 < \alpha < 1$.

By using Lemma 3.10, we have

$$\int_{\mathbb{R}^n} u^\nu f^\nu \leq C_2 (\int_{\mathbb{R}^n} f^\nu \frac{|\vec{u}|^{\frac{1}{1-\alpha}}}{(J^\nu)^{\frac{\alpha}{1-\alpha}}})^{1-\alpha},$$

which, by assumption IV gives

$$\int_{\mathbb{R}^n} u^\nu f^\nu \leq C_2 (\int_{\mathbb{R}^n} P^\nu |\vec{u}|^{s+\frac{1}{1-\alpha}} \cdot \frac{1}{(J^\nu)^{\frac{\alpha}{1-\alpha}}})^{1-\alpha} \leq C_2 [(\int_B |\vec{u}|^{s+\frac{1}{1-\alpha}} \frac{P^\nu}{(J^\nu)^{\frac{\alpha}{1-\alpha}}})^{1-\alpha} + (\int_{\mathbb{R}^n \setminus B} |\vec{u}|^{s+\frac{1}{1-\alpha}} \frac{P^\nu}{(J^\nu)^{\frac{\alpha}{1-\alpha}}})^{1-\alpha}]$$

for some constant C_2 and a ball B in $\mathbb{R}^n$ (with finite volume). By choosing $\alpha = \frac{2}{n}$ and noting that $\frac{P^\nu}{(J^\nu)^{\frac{\alpha}{1-\alpha}}} \leq C_3$ in B, we have for

$\theta = \frac{s(n-2)}{n}$,

(3.4.4) $$\int_{\mathbb{R}^n} u^\nu f^\nu(x,\vec{u}) \le C_4\|\vec{u}\|^{1+\theta}_{\frac{2n}{n-2}} + C_2\Big(\int_{\mathbb{R}^n\setminus B} |\vec{u}|^{s+\frac{1}{1-\alpha}} \frac{P^\nu}{(J^\nu)^{\frac{\alpha}{1-\alpha}}}\Big)^{1-\alpha}.$$

We note that $J^\nu \ge \frac{C}{|x|^{n-2}}$ for at least $|x|>1$, for all ν; and by choosing $\alpha = \frac{2}{n}$ again, we have

$$\Big(\int_{\mathbb{R}^n\setminus B} |\vec{u}|^{s+\frac{1}{1-\alpha}} \frac{P^\nu}{(J^\nu)^{\frac{\alpha}{1-\alpha}}}\Big)^{1-\alpha} \le C\Big(\int_{\mathbb{R}^n\setminus B} P^\nu(x)|x|^2|\vec{u}|^{s+\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}.$$

Thus

$$\begin{aligned}\Big(\int_{\mathbb{R}^n\setminus B} |\vec{u}|^{s+\frac{1}{1-\alpha}} \frac{P^\nu(x)}{(J^\nu)^{\frac{\alpha}{1-\alpha}}}\Big)^{1-\alpha} &\le C\Big(\int_{\mathbb{R}^n\setminus B} P^\nu(x)|x|^2|\vec{u}|^{(1+\theta)\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\le C\Big[\int_{\mathbb{R}^n\setminus B} (P^\nu(x)|x|^2)^{(\frac{1-\theta}{2})(\frac{n-2}{n})}\\ &\quad\times\Big(\int_{\mathbb{R}^n\setminus B} |\vec{u}|^{\frac{2n}{n-2}}\Big)^{(\frac{n-2}{2n})(1+\theta)}\Big]\\ &\le C_5\|\vec{u}\|^{1+\theta}_{\frac{2n}{n-2}}\end{aligned}$$

by using assumption V.

From (3.4.3), we obtain

$$\int_{\mathbb{R}^n} u^\nu \ell^\nu u^\nu \le C_7\|\vec{u}\|^{1+\theta}_{\frac{2n}{n-2}} + K_2.$$

By using assumption VI and Lemma 3.1, we have

$$\begin{aligned}\|u^\nu\|^2_E &\le C'\int_{\mathbb{R}^n} u^\nu \ell^\nu u^\nu\\ &\le C'_7\|\vec{u}\|^{1+\theta}_{\frac{2n}{n-2}} + C'_1\\ &\le C'_7\|\vec{u}\|^{1+\theta}_E + C'_1\end{aligned}$$

from which follows the desired estimate, since $\theta < 1$.

LEMMA 3.12. *If* $0 \leq \vec{u} \in E$ *satisfies the equation*

$$\ell^\nu u^\nu = f^\nu(x, \vec{u}) + tP^\nu(x)J^\nu \quad \text{in} \quad \mathbb{R}^n$$

for some $t \geq 0$, *then* $\vec{u} \in L^\infty(\mathbb{R}^n)$ *and* $\lim_{|x| \to \infty} u^\nu = 0$ *for* $\nu = 1, \ldots, N$.

PROOF: By making use of assumption IV, the proof follows the same arguments as in the scalar case [see Section 3.3 and Lemma 1.9], by setting

$$\vec{u}_k(x) = \left(u_k^1(x), u_k^2(x), \ldots, u_k^N(x)\right)$$

where $u_k^i(x) = \min\{u^i(x), k\}$, $i = 1, \ldots, N$ and $k = 1, 2, \ldots$.

The establishment of our existence theorem, as in Section 3.3, will follow from the verification of the conditions of Proposition E of Chapter 1. Here, the map $\Phi : Y\,(= E) \to Y$ is given by

$$\Phi(\vec{u}) = A^{-1}G(\vec{u}) \tag{3.4.5}$$

where A denotes the matrix with the operators ℓ^ν on the diagonal and zeros elsewhere and $G(\vec{u})$ is such that

$$\langle G(\vec{u}), \vec{w} \rangle = \sum_{\nu=1}^{N} \int_{\mathbb{R}^n} f^\nu(x, \vec{u}) \cdot w^\nu \, dx \tag{3.4.6}$$

for all $\vec{u}, \vec{w} \in E$. C and $\widehat{v}$ in Proposition E will be taken as

$$C = \{\vec{w} \mid w^\nu = u^+ \quad \text{for some} \quad u \in E,\ \nu = 1, \ldots, N\}$$

and

$$\widehat{v} = \begin{pmatrix} (\ell^1)^{-1}(P^1 J^1) \\ \vdots \\ (\ell^N)^{-1}(P^N J^N) \end{pmatrix} \quad \text{respectively.} \tag{3.4.7}$$

The following lemma concerns the compactness of the map Φ.

LEMMA 3.13. *Under hypothesis (I), (III), (IV) and (VI), the map* Φ *given by (3.4.5) is compact.*

PROOF: Since A^{-1} is continuous, the result will follow by showing that the map G given by (3.4.6) is compact. But this essentially follows the same arguments as in the scalar case in Section 3.3 (see also [AY]).

Using Lemma 3.13, we observe that Φ acting on C will have a fixed point $\vec{w}$ with $0 < r \leq \|\vec{w}\|_E \leq R < \infty$ provided conditions (i) and (ii) of Proposition E (in Chapter 1) are satisfied. That is

$$\beta\Phi(\vec{u}) \neq \vec{u}, \quad 0 \leq \beta \leq 1 \quad \text{for} \quad \|\vec{u}\|_E = r, \tag{3.4.8}$$

and

$$\Phi(\vec{u}) \neq \vec{u} - t\widehat{v}, \quad t \geq 0, \quad \text{for} \quad \|\vec{u}\|_E = R, \tag{3.4.9}$$

where $\widehat{v}$ is given by (3.4.7).

The following is our existence theorem:

THEOREM 3.1.4. *Suppose* f^ν $(1 \le \nu \le N)$ *satisfy (I) - (VI). Then (3.1.3) has a nontrivial decaying nonnegative solution . If in addition* $\vec{f}$ *is quasi-irreducible then each component of the solution is positive in* $\mathbb{R}^n$.

PROOF: Following the above discussion, and in view of Lemmas 3.12 and 3.13, it suffices to verify conditions (3.4.8) and (3.4.9). We start with (3.4.8). Indeed we must show that for $0 \le \beta \le 1$ and $\|\vec{u}\|_E = r$, there is no nonnegative solution of

$$\begin{aligned} \ell^\nu u^\nu &= \beta f^\nu(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n \\ \lim_{|x|\to\infty} u^\nu &= 0 \end{aligned} \tag{3.4.10}$$

if r is chosen such that $r < (\frac{K}{\beta})^{s-1}$ for some $K > 0$.

Multiplying by u^ν, integrating, and summing over ν, we have

$$\sum_{\nu=1}^{N} \int_{\mathbb{R}^n} [|\nabla u^\nu|^2 + \sum_{j=1}^{n} b_j^\nu u_{x_j}^\nu u^\nu] dx = \beta \sum_{\nu=1}^{N} \int_{\mathbb{R}^n} f^\nu \cdot u^\nu dx.$$

Thus, by using assumption (VI), we have

$$\begin{aligned} C^1 \|\vec{u}\|_E^2 &\le \beta \sum_{\nu=1}^{N} \int_{\mathbb{R}^n} f^\nu \cdot u^\nu \, dx \\ &\le \beta \sum_{\nu=1}^{N} \int_{\mathbb{R}^n} P^\nu(x) |\vec{u}|^{s+1} \\ &\le \beta K' \|\vec{u}\|_E^{s+1}, \end{aligned}$$

which implies that if $\vec{u}$ solves (3.4.10), then $\|\vec{u}\|_E \geq (\frac{K}{\beta})^{s-1}$ and (3.4.8) is established. Using Lemma 3.10 and Proposition 3.11, condition (3.4.9) follows by choosing $R > K$ (appearing in Proposition 3.11).

Analogous perturbation results to those given in Chapter 1 are possible. The procedure is the same and we forego the details.

3.5. Examples

We conclude this chapter with illustrative examples. Consider the following system

$$(3.5.1) \qquad \begin{cases} -\Delta u &= P^1(x)[u^{\alpha_1} + u^{\beta_1}v^{\beta_2} + v^{\alpha_1}] \\ -\Delta v &= P^2(x)[v^{\alpha_2} + u^{\beta_3}v^{\beta_4} + u^{\alpha_2}] \end{cases}$$

If $\beta_j \geq 0$, $j = 1, \ldots, 4$ with

$$1 < \beta_1 + \beta_2 = \alpha_1 < \frac{n}{n-2}, \quad 1 < \beta_3 + \beta_4 = \alpha_2 < \frac{n}{n-2},$$

and $0 \leq P^i(x) = O(|x|^{-a})$ at ∞, $a > n$ with $P^i(x)|x|^2 \in L^{\frac{2}{1-\theta}}(\mathbb{R}^n)$, $\theta = \frac{\alpha_i(n-2)}{n}$, $i = 1, 2$; then by Theorem 3.14 there exists positive decaying solution (u, v) of system (3.5.1). Comparing with the examples in Chapter 1, we observe that we need not require monotonicity conditions on the functions P^1, P^2 near ∞ and these functions are also now allowed to be merely nonnegative.

Consider now the system

$$(3.5.2)\qquad \begin{cases} -\Delta u + \sum_{j=1}^{n} b_j(x) D_j u &= P^1(x)[u^{\alpha_1} + u^{\beta_1} v^{\beta_2} + v^{\alpha_1}] \\ -\Delta v + \sum_{j=1}^{n} b_j(x) D_j v &= P^2(x)[v^{\alpha_2} + u^{\beta_3} v^{\beta_4} + u^{\alpha_2}] \end{cases}$$

where P^i, α_i, $i = 1,2$ and β_j, $j = 1,\ldots,4$ are as above. If $0 \not\equiv b_j \in C^1 \cap L^\infty$ with $-\sum D_j(b_j) \geq 0$ and there exists a constant K_1 such that $\sum |\nabla b_j|^2 + \sum_{j=1}^{n} |b_j(x)|^2 \leq K_1 \sigma^{2+\epsilon}(x)$, then (3.5.2) has a positive decaying solution. Observe that the condition $-\sum D_j(b_j) \geq 0$ on b_j guarantees that assumption (VI) holds.

CHAPTER 4

SEMILINEAR ELLIPTIC SYSTEMS WITH VARIATIONAL STRUCTURE

4.1. Introduction

In this chapter, we consider the existence of positive solutions for parametrized weakly coupled semilinear elliptic systems (with variational structure) of the type

$$\text{(4.1.1)} \qquad \begin{cases} \ell u^i = \lambda f^i(x, \vec{u}) & \text{in } \mathbb{R}^n \\ \lim\limits_{|x|\to\infty} u^i = 0, & i = 1, \dots, N, \end{cases}$$

where $n \geq 3$, $\lambda > 0$ and ℓ denotes the uniformly elliptic expression:

$$\ell u^k = -\sum D_i\big(a_{ij}(x) D_j u^k\big) + b(x) u^k,$$

with $a_{ij} = a_{ji} \in C^{1+\alpha}_{\text{loc}} \cap L^\infty(\mathbb{R}^n)$, $0 \leq b \in C^{\alpha}_{\text{loc}} \cap L^\infty(\mathbb{R}^n)$. System (4.1.1) is said to have variational structure if the elliptic operator ℓ has variational form and the nonlinearities f^i are such that there exists a differentiable function F with respect to the components of $\vec{u}$ with

$$\text{(4.1.2)} \qquad \nabla_{\vec{u}} F(x, \vec{u}) = \big(f^1(x, \vec{u}), f^2(x, \vec{u}), \dots, f^N(x, \vec{u})\big).$$

We admit that this is a strong requirement as noted in Chapter 1; however our interest is to extend some of the earlier results obtained in this case. As in the

previous chapters, the functions f^i will be assumed to be superlinear and subcritical at ∞. Precise conditions will be given below.

For scalar equations, $N = 1$, we mentioned in Chapter 3 some of the results obtained for the radial case with $b(x) \equiv \widetilde{b}$ (a positive constant). In the general nonradial cases for such equations with $b(x) \geq b_0 > 0$, we mention here the paper by Lions [Li2], where the so-called "concentration-compactness" method was used. The case in which the coefficient function $b(x)$ is allowed to be zero, i.e., $b(x) \geq 0$ was also investigated. Here, Kawano, Sutsuma and Yotsutani [KSY] and Li and Ni [LN1][LN2] obtained results for the radially symmetric nonlinearities; and Allegretto and Yu [AY], Noussair and Swanson [NS4][NS5] studied the nonradial case. The results in [AY], [NS4], [NS5] were given for equations of the type:

$$-\Delta u + b(x)u = \lambda f(x,u) \quad \text{in} \quad \mathbb{R}^n \tag{4.1.3}$$

with $\lambda > 0$. In [NS4][NS5], the authors obtained the existence of a positive solution, for some $\lambda = \lambda_0 > 0$, of (4.1.3) under the assumption

$$0 \leq f(x,u) \leq f_1(x)u^\alpha$$

with $0 \leq f_1 \in L^q(\mathbb{R}^n)$, $1 < q < \frac{2n}{2n-(n-2)(\alpha+1)}$ and $1 < \alpha < \frac{n+2}{n-2}$ or $\frac{(n+2)q-2n}{(n-2)q} < \alpha < \frac{n+2}{n-2}$ for a given q. Their method involved applying variational techniques in the standard Sobolev space $W_0^{1,2}(\mathbb{R}^n)$ to a sequence of

approximating equations and then proving that the resulting sequence of solutions converges to the desired solution of (4.1.3). In [AY], Allegretto and Yu on the other hand used variational method via variants of the Mountain Pass theorem and weighted spaces and gave conditions that guarantee the existence of positive decaying solutons of (4.1.3) for any λ and not just some λ.

The first existence result for system (4.1.1) (assumed to have variational structure) in the case of autonomous nonlinearities, with $\lambda = 1$ was stated by H. Berestycki and P.L. Lions [BL2]. The details were however not given. For this case also, H. Brezis and E.H. Lieb [BrL] considered the minimum action solutions — without explicitly discussing the existence of positive solutions. Only very recently, E.S. Noussair and C.A. Swanson [NS3] considered the question of existence of positive solutions for systems of the type (4.1.1) — assuming that the system has variational structure. They obtained existence results for positive decaying solutions in the case in which $f^i(x, \vec{u})$ is assumed to satisfy a power growth condition in the components of $\vec{u}$, with exponents $\beta = (\beta^1, \ldots, \beta^N)$ of each power of $\vec{u}$ restricted by $1 < \sum_{i=1}^{N} \beta^i < \frac{n+2}{n-2}$. Their results were however given only for some parameter $\lambda_0 > 0$ and not for any $\lambda > 0$ and for small N. The authors employed the variational method to a sequence of approximating systems of equations and then proved that the resulting sequence of approximating solutions converges to the desired solution.

Our aim in this chapter is to give conditions that will guarantee the existence of positive solution for sytem (4.1.1), for any $\lambda > 0$ and not just some

λ. Furthermore, while in general the maximum N depends on n, we allow $N = 2$ for any n. This essentially extends the result of Noussair and Swanson. Our method involves a direct application of the variational method to system (4.1.1) — with our main tools being weighted spaces and the Mountain Pass Theorem.

This chapter is organized as follows: following this section on introduction, we proceed in Section 4.2 to recall some of the weighted spaces we will need in this chapter, present our assumptions on the nonlinearities f^i, give some preliminary results and state some abstract results — such as the Mountain Pass Theorem that we will need in this and subsequent chapters. In Section 4.3, we present our existence result. We prove in particular that the variational system (4.1.1) has a positive solution for any parameter $\lambda > 0$. Finally in Section 4.4, the chapter concludes with illustrative examples which compare our results to earlier ones.

4.2. Assumptions and Preliminary Results

In this section, we state our assumptions on the nonlinearities f^i, present some preliminary results and state the Mountain Pass Theorem. Due to the lack of Poincaré-type inequalities in $\mathbb{R}^n$, we cannot use the usual Sobolev space $W_0^{1,2}(\mathbb{R}^n)$ because of the assumption $b(x) \geq 0$ on the coefficient function. Thus we start by recalling the weighted space E introduced in Chapters 2

and 3 above, with the norm given by

$$\|\vec{u}\|_E = \Big[\sum_{i=1}^{N} \|u^i\|_E^2\Big]^{1/2}. \tag{4.2.1}$$

Here however, $\|\cdot\|_E$ will be given by

$$\|v\|_E = \Big(\int_{\mathbb{R}^n} \|\nabla v\|^2 + \tilde{\sigma} v^2\Big)^{1/2} \quad \text{for} \quad v \in E,$$

where $\tilde{\sigma}(x) = \max\{\sigma(x), b(x)\}$. Recall that $\sigma(x) = (1+|x|^2)^{-1}$. We observe as in [AY] that

$$\|\vec{u}\|_\ell^2 =: (\ell\vec{u},\ \vec{u}) = \sum_{i=1}^{N} (\ell u^i, u^i) \tag{4.2.2}$$

formally defines a norm on E which, by Hardy's Inequality, is equivalent to $\|\cdot\|_E$. We observe also that we can identify, locally, $E\times\cdots\times E$ with the usual Sobolev space $[H^{1,2}(\mathbb{R}^n)]^N = H^{1,2}(\mathbb{R}^n)\times\cdots\times H^{1,2}(\mathbb{R}^n)$ (N multi product). Thus, sequences bounded in $\|\cdot\|_\ell$ have locally convergent subsequences in $[L^2(\mathbb{R}^n)]^N$. Finally, $\langle\cdot,\cdot\rangle_N$ will denote the E inner product induced by $\|\cdot\|_\ell$.

We now state our assumptions on f^i:

(A1) $f^i \in C^\alpha_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^N, \mathbb{R})$, $\alpha \in (0,1)$, $i = 1,\dots,N$, $f^i(x,\vec{u}) \geq 0$ for $x \in \mathbb{R}^n$, $\vec{u} \geq \vec{0}$; with $f^i(x,\vec{u}) > 0$ if every $u^j > 0$, $j = 1,\dots,N$.

(A2) $|f^i(x,\vec{t})| \leq f_1(x)|\vec{t}|^s$, $x \in \mathbb{R}^n$, $\vec{t} \in \mathbb{R}^N$, with $0 < f_1 \in L^{p_0} \cap L^\infty(\mathbb{R}^n)$, $1 < s < \frac{n+2}{n-2}$, $p_0 = \frac{2n}{\big(2n-(s+1)(n-2)\big)}$.

(A3) $f^k(x,\vec{t}\,)$ is continuously differentiable with respect to the components of $\vec{t}$ and

$$\frac{\partial f^i(x,\vec{t}^{+}\,)}{\partial t^j} = \frac{\partial f^j(x,\vec{t}^{+})}{\partial t^i} \quad \text{for} \quad x \in \mathbb{R}^n, \quad \vec{t} \in \mathbb{R}^N$$

$$i,j = 1,2,\ldots N, \quad i \neq j.$$

We will, without fear of confusion since we seek positive solutions and for notational convenience, write $f^i(x,\vec{t})$ in place of $f^i(x,\vec{t}^{+})$ in the sequel.

(A4) There exists a constant $\theta > 0$ with $\frac{\theta}{N} < \frac{1}{2}$ such that

$$\theta u^i f^i(x,\vec{u}) \geq F(x,\vec{u}) \quad \text{for} \quad \vec{u} \geq 0, \quad x \in \mathbb{R}^n,$$

$$i = 1,\ldots,N;$$

where $F(x,\vec{u})$ is the differentiable function satisfying

$$\nabla_{\vec{u}} F(x,\vec{u}) = \big(f^1(x,\vec{u}),\ldots,f^N(x,\vec{u})\big) =: \vec{f}(x,\vec{u}),$$

i.e., $F(x,\vec{u})$ is the potential associated with $\vec{f}(x,\vec{u})$, which exists in view of assumption (A3). We shall take $F(x,\vec{0}) = 0$.

(A5) There exists $w \in C_0^\infty$, $w \geq 0$, $w \not\equiv 0$ such that for all $\beta \geq 0$

$$\int_{\mathbb{R}^n} F(x,\beta w\vec{1}) \geq \beta^\mu C(w) - K(w).$$

Here $\mu > 2$, $K(w)$, $C(w)$ are constants depending on w, and $\vec{1} = (1, \ldots, 1)$.

REMARK: Assumption (A4) implies $\vec{f}$ is quasi-irreducible. Indeed assume $\vec{u} \geq \vec{0}$, $\vec{u} \not\equiv \vec{0}$, satisfies (4.1.1). If $u^j \equiv 0$ for some j then $f^i(x, \vec{u}(x)) \equiv 0$ for all i (whence $\vec{u} \equiv \vec{0}$). To see this, assume $f^i(x_0, \vec{u}(x_0)) \neq 0$ for some i, $x_0 \in \mathbb{R}^n$. Then $u^i(x_0) > 0$ and

$$\begin{aligned} 0 &= \theta u^j(x_0) f^j(x_0, \vec{u}(x_0)) \\ &\geq F(x_0, \vec{u}(x_0)) \\ &= \int_0^1 \sum_k f^k(x_0, t\vec{u}(x_0)) \cdot u^k(x_0) dt \\ &> 0 \end{aligned}$$

since $f^k \geq 0$. This is a contradiction. It therefore follows that $u^j \geq 0$, $u^j \not\equiv 0$ and hence $u^j > 0$ for all j by the scalar maximum principle. It would be thus possible to replace assumption (A4) by assuming quasi-irreducibility; we may indeed assume instead that the problem is quasi-irreducible and $F(x, \vec{u}) \leq \theta \sum f^i(x, \vec{u}) u^i$, with $\theta < \frac{1}{2}$ – an assumption which will enable us treat more general nonlinearities, but in view of the results in [NS3] we keep condition (A4).

We let $G(\vec{u}) = \int_{\mathbb{R}^n} F(x, \vec{u}) dx$ and

$$J(\vec{u}) = \tfrac{1}{2} \|\vec{u}\|_\ell^2 - \lambda G(\vec{u}) \quad \text{for} \quad \vec{u} \in E.$$

LEMMA 4.1. *(a) G and J are weakly lower semi-continuous with*

$$G'(\vec{u})(\vec{\varphi}) = \int_{\mathbb{R}^n} \vec{f}(x,\vec{u}) \cdot \vec{\varphi}\, dx.$$

(b) G' is continuous and compact from E to E.

PROOF: First, we note that

$$\begin{aligned} |G(\vec{u})| &\le C' \int_{\mathbb{R}^n} f_1 |\vec{u}|^{s+1} dx \\ &\le C' \|f_1\|_{L^{p_0}} \|\vec{u}\|^{s+1}_{\frac{2n}{n-2}} \\ &\le C \|f_1\|_{L^{p_0}} \|\vec{u}\|^{s+1}_{\ell} \end{aligned}$$

by the Sobolev's Inequality; which implies that G is a well-defined functional on E. By using the properties of the norm $\|\cdot\|_\ell$, the proof of part (a) essentially follows the same arguments as in the scalar case [AY] (see also [NS3]). As for part (b), we observe that for each $\vec{u} \in E$, $G'(\vec{u})$ is a linear functional on E. Since E is a Hilbert space, $G'(\vec{u})$ is an element of E. Thus by choosing $\vec{0} \neq \vec{\varphi} = G'(\vec{u}_n) - G'(\vec{u})$ we have

$$\begin{aligned} \|G'(\vec{u}_m) - G'(\vec{u})\|^2_\ell &= \sum_i \int_B [f^i(x,\vec{u}_m) - f^i(x,\vec{u})] \cdot \big(G'(u^i_m) - G'(u^i)\big) \\ &\quad + \sum_i \int_{\mathbb{R}^n \setminus B} [f^i(x,\vec{u}_m) - f^i(x,\vec{u})\big(G'(u^i_m) - G'(u^i)\big) \\ &\le C[\sum_i \|f^i(\cdot,\vec{u}_m) - f^i(\cdot,\vec{u})\|_{L^{\frac{2n}{n+2}}(B)} \\ &\quad + \|f_1\|_{L^{p_0}(\mathbb{R}^n \setminus B)}(\|\vec{u}_m\|^s_\ell + \|\vec{u}\|^s_\ell)] \cdot \|G'(\vec{u}_m) - G'(\vec{u})\|_\ell, \end{aligned}$$

from this, we have

$$\|G'(\vec{u}_m) - G'(\vec{u})\|_\ell \le C[\sum_i \|f^i(\cdot, \vec{u}_m) - f^i(\cdot, \vec{u})\|_{L^{\frac{2n}{n+2}}(B)} + \|f_1\|_{L^{p_0}(\mathbb{R}^n \setminus B)}(\|\vec{u}_m\|_\ell^* + \|\vec{u}\|_\ell^*)],$$

for some ball $B \subset \mathbb{R}^n$. We may thus argue as in the scalar case [AY] to conclude that G' is continuous and compact from E to E.

In view of Lemma 4.1, the weak solutions $\vec{u}$ of (4.1.1) in E may thus be described as the critical points of J. For such solutions, we have:

LEMMA 4.2. *(a)* $\vec{u} \in L^p(\mathbb{R}^n)$, $\frac{2n}{n-2} \ge P \le \infty$.

(b) $\lim_{|x| \to \infty} u^k = 0$ *for* $k = 1, 2, \ldots, N$.

PROOF: We set $\vec{u}_k(x) = (u_k^1(x), u_k^2(x), \ldots, u_k^N(x))$ where $u_k^i(x) = \min\{u^i(x), k\}$, $i = 1, \ldots, N$ and $k = 1, 2, \ldots$. For any real number $m \ge 1$, let

$$[\vec{u}_k(x)]^m = ([u_k^1(x)]^m, \ldots, [u_k^N(x)]^m).$$

The solution $\vec{u}$ being a critical point of J satisfies

$$J'(\vec{u})(\vec{\varphi}) = \langle \vec{u}, \vec{\varphi} \rangle_N - \lambda \int_{\mathbb{R}^n} \vec{f}(x, \vec{u}) \cdot \vec{\varphi}\, dx = 0$$

for $\vec{\varphi} \in E$. Taking $\vec{\varphi}(x) = [\vec{u}_k(x)]^m$, we have

$$\langle \vec{u}, [\vec{u}_k(x)]^m \rangle_N \leq \lambda \int_{\mathbb{R}^n} \sum_{j=1}^{N} |f^j(x, \vec{u})| [u_k^j]^m dx$$

$$\leq \sum_{j=1}^{N} \lambda \int_{\mathbb{R}^n} |f^j(x, \vec{u})| [u^j]^m dx.$$

But

$$\langle \vec{u}, [\vec{u}_k(x)]^m \rangle_N \geq \sum_{j=1}^{N} \frac{4m}{(m+1)^2} \| [u_k^j]^{\frac{(m+1)}{2}} \|^2_{-\Delta}$$

$$\geq \sum_{j=1}^{N} C(n,m) \left[\|u_k^j\|_{\frac{(m+1)n}{n-2}} \right]^{m+1}.$$

Thus

$$\sum_{j=1}^{N} C(n,m) \left[\|u_k^j\|_{\frac{(m+1)n}{n-2}} \right]^{m+1} \leq \sum_{j=1}^{N} \lambda \int_{\mathbb{R}^n} |f^j(x, \vec{u})| [u^j]^m dx,$$

and the proof follows as in the scalar case (see [AY]).

Finally in this section, we present the following abstract result — the Mountain Pass Theorem of Ambrosetti and Rabinowitz. This theorem has proven to be a powerful tool — used to obtain existence of solutions for superlinear and subcritical problems via the variational method approach. We shall essentially employ this theorem in the next section to obtain our existence result. The statement we give here is that found for example, in [Str]. Let E be a real Banach Space, $C^1(E, \mathbb{R})$ will denote the set of functionals that are Fréchet differentiable and whose Fréchet derivatives are continuous on E.

DEFINITION: (a) A sequence $\{u_m\}$ in E is a Palais-Smale sequence for $J \in C^1(E, \mathbb{R})$ if $|J(u_m)| \leq C$, uniformly in m, while $\|J'(u_m)\| \to 0$ strongly, as $m \to \infty$.

(b) $J \in C^1(E, \mathbb{R})$ is said to satisfy the Palais-Smale condition (denoted (P.-S.) condition) if any Palais-Smale sequence for J has a (strongly) convergent subsequence.

THEOREM A (MOUNTAIN PASS THEOREM). *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy (P.-S.) condition. Suppose*

(a) $J(0) = 0$

(b) there exists $\rho > 0$, $\alpha > 0$ *such that* $\|u\| = \rho$ *implies*

$$J(u) \geq \alpha;$$

(c) there exists $u_1 \in E$ *such that* $\|u_1\| \geq \rho$ *and*

$$J(u_1) < \alpha.$$

Define

$$P = \{p \in C([0,1], E) \mid p(0) = 0,\ p(1) = u_1\}.$$

Then

$$\beta = \inf_{p \in P} \sup_{u \in p} J(u)$$

is a critical value.

The next proposition gives conditions that guarantee that a functional J satisfies the (P.-S.) condition.

PROPOSITION B. *Suppose* $J \in C^1(E, \mathbb{R})$ *on a Banach space* E. *Suppose that* J *has the property that*

(a) Any Palais-Smale sequence for J *is bounded in* E.

Further assume that

(b) for any $u \in E$ *we can decompose*

$$J'(u) = L + K(u),$$

where $L : E \to E^*$ *(the dual space) is a fixed boundedly invertible linear map and the operator* K *maps bounded sets in* E *to relatively compact sets in* E^*. *Then* J *satisfies (P.-S.) condition.*

The proofs of the theorem and the proposition can be found, for example, in [Str]. See also [Ra].

4.3. Existence Result

In this section, we present our existence result.

THEOREM 4.3. *If (A1)–(A4) hold, then for any parameter* $\lambda > 0$ *the system (4.1.1) has at least one positive decaying solution* $\vec{u} \in L^p(\mathbb{R}^n)$ *for* $\frac{2n}{n-2} \leq p \leq \infty$.

PROOF: The proof of the theorem is essentially the verification of the conditions of Theorem A. We set $f^k(x,u^1,\dots,u^i,\dots,u^N) = f^k(x,u^1,\dots,0,\dots,u^N)$ if $u^i < 0$ and $u^j \geq 0$ for $j \neq i$, $k = 1,\dots,N$. Here, our Banach space is E and our functional (as stated in Section 4.2) is given by

$$J(\vec{u}) = \tfrac{1}{2}\,\|\vec{u}\|_\ell^2 - \lambda \int_{\mathbb{R}^n} F(x,\vec{u})dx, \quad \vec{u} \in E.$$

The fact that $F(x,\vec{0}) = 0$ implies that $J(\vec{0}) = 0$, which satisfies condition (a) of Theorem A. Next we obsereve that

$$\begin{aligned} J(\vec{u}) &= \tfrac{1}{2}\,\|\vec{u}\|_\ell^2 - \lambda \int_{\mathbb{R}^n} F(x,\vec{u})dx \\ &\geq \tfrac{1}{2}\,\|\vec{u}\|_\ell^2 - \lambda C \int_{\mathbb{R}^n} f_1 |\vec{u}|^{s+1} dx \\ &\geq \tfrac{1}{2}\,\|\vec{u}\|_\ell^2 - \lambda C \|f_1\|_{L^{p_0}} \|\vec{u}\|_\ell^{s+1} \\ &= \left(\tfrac{1}{2} - \lambda C \|f_1\|_{L^{p_0}} \|\vec{u}\|_\ell^{s-1}\right) \|\vec{u}\|_\ell^2. \end{aligned}$$

Choosing $\rho > 0$ sufficiently small, we can find $\alpha > 0$ such that $\|\vec{u}\|_\ell = \rho$ implies $J(\vec{u}) \geq \alpha$ — satisfying condition (b). From assumption (A5), we have for $\beta > 0$

$$\begin{aligned} J(\beta w\vec{1}) &= \tfrac{1}{2}\,\|\beta w\vec{1}\|_\ell^2 - \lambda \int_{\mathbb{R}^n} F(x,\beta w\vec{1})dx \\ &\leq \tfrac{1}{2}\,\beta^2 \|w\vec{1}\|_\ell^2 - \lambda\beta^\mu C(w) + \lambda K(w). \end{aligned}$$

From this inequality, we conclude that $J(\beta w\vec{1}) < 0$ for β large enough — which verifies condition (c) of Theorem A. Lemma 4.1 and standard procedure

show that $J \in C^1(E, \mathbb{R})$. Finally we show that J satisfies the (P.-S.) condition. This however follows from the compactness of $G'(\cdot)$ (see Lemma 4.1), in view of Theorem B and the fact that

$$J'(\vec{u})(\vec{\varphi}) = \langle \vec{u}, \vec{\varphi} \rangle_N - \lambda \int_{\mathbb{R}^n} \vec{f}(x, \vec{u}) \cdot \vec{\varphi}\, dx.$$

So all the conditions of Theorem A are satisfied and we conclude that $J(\cdot)$ has a critical point, say $\vec{u}$. From assumptions, it is seen that $\vec{u} \geq 0$ a.e. with $\vec{u} \not\equiv \vec{0}$. That $u^i > 0$ for $i = 1, \ldots, N$ is a consequence of the fact that our problem is quasi-irreducible by the remark following assumption (A5). $\vec{u} \to \vec{0}$ follows from Lemma 4.2 and the proof is complete.

4.4. Comparison and Examples

We conclude this chapter with illustrative examples which compare our result to earlier ones.

EXAMPLE 1: Consider the system

$$(4.4.1) \qquad \begin{cases} -\Delta u &= \lambda \alpha_3 h(x) u^{\alpha_1} v^{\alpha_2} \\ -\Delta v &= \lambda \alpha_2 h(x) u^{\alpha_3} v^{\alpha)4} \end{cases} \quad \text{in} \quad \mathbb{R}^n.$$

Assume $\alpha_i \geq 0$ for $i = 1, \ldots, 4$, $\alpha_4 + 1 = \alpha_2$, $\alpha_1 + 1 = \alpha_3$, $1 < \alpha_1 + \alpha_2 < \frac{n+2}{n-2}$ and $0 < h(x) = O(|x|^{-a})$, $a > 0$ with $\max\{\frac{(n+2-2a)}{(n-2)}, 1\} < \alpha_1 + \alpha_2$. Observe that $h \in L^{p_0}(\mathbb{R}^n)$, $p_0 = \frac{2n}{\left(2n-(\alpha_1+\alpha_2+1)(n-2)\right)}$ since $ap_0 > n$, and by Theorem 4.3, system (4.4.1) has a positive decaying solution $\big(u(x), v(x)\big)_\lambda$ for any parameter $\lambda > 0$. Systems of the type (4.4.1) have been considered

in [NS3] under somewhat similar conditions as above. The authors obtained existence of positive decaying solutions only for some $\lambda = \tilde{\lambda}$ and not for any λ and for $N \leq 5$.

We remark that by arguing as above and making suitable changes on the conditions on f_1 and the space E, as in Chapter 2, we are able to obtain similar existence theorem as Theorem 4.3 for the elliptic system

$$-\Delta u^i + u^i = \lambda f^i(x, \vec{u}) \quad \text{in} \quad \mathbb{R}^n, \tag{4.4.2}$$

where f^i $(1 \leq i \leq N)$ continue to satisfy all the assumptions stated in Section 4.2. We however observe that E is now taken as the usual Sobolev space $H^{1,2}$ and the condition $f_1 \in L^{p_0} \cap L^\infty$ is replaced with $f_1 \in L^\infty$ and

$$M_{\alpha,1}(f_1, \mathbb{R}^n \backslash B_R) \to 0 \quad \text{as} \quad R \to \infty$$

(refer to Chapter 2). We have:

EXAMPLE 2: Consider

$$\begin{cases} -\Delta u + u &= \lambda \alpha_3 q(x) u^{\alpha_1} v^{\alpha_2} \\ -\Delta v + v &= \lambda \alpha_2 q(x) u^{\alpha_3} v^{\alpha_4} \end{cases} \quad \text{in} \quad \mathbb{R}^n. \tag{4.4.3}$$

Let α_i be as in Example 1 and assume that $0 < q \in L^\infty$ and $\|q\|_{L^1(B_1(x))} \to 0$ as $|x| \to \infty$, then the conclusion of Example 1 holds for system (4.4.3). Compared with Example 2 of Chapter 2, we observe that we do not require q to satisfy any monotonicity conditions. However, system (4.4.3) is variational.

CHAPTER 5

NONPOSITONE ELLIPTIC PROBLEMS IN $\mathbb{R}^n$

5.1. Introduction

In all of the previous chapters, we have assumed that the nonlinearities f^i satisfy: $f^i(x,\vec{0}) \geq 0$ for $x \in \mathbb{R}^n$, $i = 1,\ldots,N$. This assumption allows, in general, for the system of equations under consideration to admit the trivial solution or for the trivial solution to be a sub-solution. In this chapter, we consider the case in which $f^i(x,\vec{0}) \leq 0$ with $f^i(x,\vec{0}) \not\equiv 0$ for $x \in \mathbb{R}^n$ and $1 \leq i \leq N$ (nonpositone). We start with the case $N = 1$ – that is, the scalar case.

The theory of elliptic nonpositone problems has received considerable attention since its inception in the symmetry breaking considerations of Smoller and Wasserman in the mid 1980's [SW1], [SW2], [SW3]. We recall in particular the papers of Allegretto, Nistri and Zecca [ANZ], Brown, Castro and Shivaji [BCS], Brown and Shivaji [BSh], Castro and Shivaji [CS1], [CS2] and the thesis of Unsurangsie [Un], where existence, nonexistence and stability are discussed. The prototype equation considered is of the following type:

$$(5.1.1) \qquad \begin{cases} \ell u \equiv -\sum D_i\big(a_{ij}(x)D_j u\big) = \lambda f(x,u) & \text{in } \Omega \\ u \equiv 0 & \text{on } \partial\Omega \end{cases}$$

with $\lambda > 0$, ℓ a uniformly elliptic operator in the smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and the problem is to obtain conditions for the existence of

a positive solution. As indicated above, what distinguishes nonpositone problems given by (5.1.1) from the more typical nonlinear equation of the same form is the assumption: $f(x,0) \leq 0, \quad f(x,0) \not\equiv 0$. In [SW3], Smoller and Wasserman considered the autonomous nonlinearity with the domain Ω taken to be an n-ball and $\ell = -\Delta$. They proved an existence result applicable to functions of the type $f(u) = u^s - \varepsilon$ where $\varepsilon > 0$, $1 < s < \frac{n}{n-2}$. Still considering the autonomous nonlinearity, $\ell = -\Delta$ and Ω a ball, Castro and Shivaji [CS2] extended this result to cover the case in which the subcritical growth exponent s covers the "full" range - that is, $1 < s < \frac{n+2}{n-2}$. They proved, in particular, that if $f : [0,\infty) \to \mathbb{R}$ is such that $f' \geq 0, \quad f(0) < 0$ (nonpositone) $\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty$ (f is superlinear) and f satisfies some technical conditions, then there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, (5.1.1) has a positive, decreasing solution in Ω. The authors employed the O.D.E. approach - based on the shooting method. In [Un], Unsurangsie utilized the Mountain Pass arguments and suitable estimates to obtain similar results in the case of a general smooth bounded domain Ω. The first known consideration of the non-autonomous nonlinearity for nonpositone problems in a general smooth bounded domain was by Allegretto, Nistri and Zecca [ANZ]. In this paper, the authors investigated the case of a more general elliptic operator ℓ, given by

$$\ell u = -\Delta u + 2 \sum b_j D_j u$$

with $b_j \in C^\alpha(\overline{\Omega})$, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain. Existence results were obtained for λ small, in the case in which f is superlinear and subcritical – that is, $f(x,\xi) \sim \xi^s$ for $0 < \xi$ large, $1 < s < \frac{n+2}{n-2}$. The main tools used by the authors were a priori estimates and Degree Theory. Nonexistence results have been obtained in [BSC] and [ANZ], and stability results in [BSh]. To the best of our knowledge, all the previous considerations have dealt with the case of a bounded domain Ω and scalar equations.

In this chapter, we consider the case $\Omega = \mathbb{R}^n$, $u \to 0$ at ∞; and thus we must deal with the resulting lack of compactness of the embedding operators between the canonical Sobolev spaces. We shall assume that f is superlinear and subcritical at ∞. Our existence proof will consequently rely on the use of Weighted Spaces and Mountain Pass arguments (see Theorems A and B of Chapter 4) as in the positone case, [AY]. We shall also consider the nonpositone problem for the following semilinear elliptic system:

$$(5.1.2) \qquad \begin{cases} \ell u^k = -\sum_{i,j} D_i\big(a_{ij}(x) D_j u^k\big) = \lambda f^k(x,\vec{u}) & \text{in } \mathbb{R}^n \\ u^k \to 0 \quad \text{as } |x| \to \infty, & k = 1,2,\ldots,N \end{cases}$$

where $n \geq 3$, $x = (x_1,\ldots,x_n)$, $\vec{u} = (u^1,u^2,\ldots,u^N)$ and $\lambda > 0$. The conditions on f^k will be specified below.

This chapter is organised as follows: after this introduction, we proceed, in Section 5.2, to recall some useful weighted spaces, state some of our assumptions and give some preliminary results. In Section 5.3, we present our first main result. For completeness and convenience in later proofs, we first briefly sketch

how the existence of a solution u (not necessarily positive) of (5.1.1), for λ small enough, follows by Mountain Pass arguments. Next, we consider the question of showing the positivity of such a u, and observe that we cannot use the Maximum Principle directly to guarantee $u > 0$. Instead we replace this with estimates on various norms of u in various subsets of Ω and with arguments based on the special case $a_{ij} = \delta_{ij}$, that is, for $\ell = -\Delta$. Theorems on comparisons of Green's functions of Stampacchia, [St], are then recalled to deal with the general case. In Section 5.4, we use an identical approach to extend the results obtained in Section 5.3 for (5.1.1) to system (5.1.2). That is, we obtain existence result for positive solutons of (5.1.2) under the assumption that the nonlinearities f^k satisfy the nonpositone condition, stated in the first paragraph of this section, among others. We conclude this chapter with illustrative examples in Section 5.5.

5.2. Assumptions and Preliminary Results

We assume for simplicity that $a_{ij} \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$, $f \in C^{\alpha}_{\text{loc}}(\mathbb{R}^{n+1})$ and that the symmetric matrix (a_{ij}) satisfies $aI \leq (a_{ij}(x)) \leq bI$ for some positive constants α, a, b.

We recall the weighted space E, as in Chapter 3, as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_0^2 = \int_{\mathbb{R}^n} |\nabla u|^2 + \sigma u^2 dx$$

with $\sigma = (1+|x|^2)^{-1}$ and further recall from [AY] the observation that $\|u\|_0 \sim \|u\|_\ell$ where $\|u\|_\ell^2 = (\ell u, u)$ as a consequence of Hardy's Inequality. As in the previous chapters, the various norms of the vector functions $\vec{u}$ are defined in terms of the norms of the components of $\vec{u}$ in obvious ways; and vector inequalities will be understood to hold componentwise.

We next observe that we may always write $f(x,t) \triangleq g(x,t)+f(x,0)$ and take, without loss of generality, $g(x,t) \equiv 0$ if $t \leq 0$. As mentioned earlier, we deal with the subcritical case; specifically, we assume that:

(I) $|g(x,t)| \leq f_1(x)t^s$, $|f(x,0)| \leq f_0(x)$, with $t \geq 0$, $f_0 \in L^2_{\sigma^{-1}} \cap L^\infty$, $f_1 \in L^{p_0} \cap L^\infty(\mathbb{R}^n)$, $1 < s < \frac{n+2}{n-2}$, $p_0 = \frac{2n}{[2n-(s+1)(n-2)]}$.

Here $L^2_{\sigma^{-1}}$ denotes the L^2 space with weight σ^{-1}.

Further assumptions on f, and similar observation and assumption on the f^ks will be made below.

We let: $G(u) = \int_{\mathbb{R}^n} F(x,u)dx$, $F(x,u) = \int_0^u f(x,\xi)d\xi$, $J(u) = \frac{1}{2}\|u\|_\ell^2 - \lambda G(u)$ for $u \in E$ and recall:

LEMMA 5.0. *(a) G and J are weakly lower semi-continuous with $G'(u) = \int_{\mathbb{R}^n} f(x,u)\varphi dx$.*

(b) G' is continuous and compact from E to E.

The explicit proof of these statements may be found, for example, in [AY]. The weak solutions u of (5.1.1) in E may thus be described as the critical points of J, and we finally recall for such solutions:

LEMMA 5.1. *(a)* $u \in L^p(\mathbb{R}^n)$, $\frac{2n}{n-2} \leq p \leq \infty$,

(b) $\lim_{|x| \to \infty} u = 0.$

Once again the proof given in [AY] or the earlier chapters still holds.

5.3. Existence Result for (5.1.1)

We first show that (5.1.1) has a solution and estimate its norm. Let us express $F(x,u) = F_1(x,u) + F_2(x,u)$ with

$$F_1(x,u) = \int_0^u g(x,t)dt \quad \text{and} \quad F_2(x,u) = f(x,0)u.$$

We now assume that

(II) $F_1(x,\alpha_0) > 0$ for some constant $\alpha_0 > 0$ and x in some open set Ω_0 and that for $\mu = s+1$ we have

$$\mu F_1(x,t) \leq tg(x,t) \quad \text{for} \quad x \in \mathbb{R}^n, \quad t \geq 0.$$

THEOREM 5.2. (EXISTENCE). *Under the above assumptions (I), (II) on f, problem (5.1.1) has for λ small a solution $u \in E$ such that*

$$K\lambda^{\frac{-1}{(s-1)}} \leq \|u\|_\ell \leq H\lambda^{\frac{-1}{(s-1)}},$$

for some constants K, H independent of u.

PROOF: The proof is an adaptation of ideas from [AY] and [Un] with the existence part basically an application of the Mountain Pass Theorem (Theorem A

in Chapter 4). We first observe that the P.-S. condition is satisfied. Indeed, assume $\{u_m\}$ is a sequence with $\{J(u_m)\}$ bounded and $J'(u_m) \to 0$. In view of Lemma 5.0 and Theorem B in Chapter 4, it suffices to show that $\|u_m\|_\ell$ is bounded; and to see this, we observe:

$$\begin{aligned} J'(u_m)(u_m^-) &= -\int_{\mathbb{R}^n} \sum a_{ij}(u_m^-)_i(u_m^-)_j - \lambda \int_{\mathbb{R}^n} \left(g(x,u_m)u_m^- + f(x,0)u_m^-\right) \\ &= -\left[\int_{\mathbb{R}^n} \sum a_{ij}(u_m^-)_i(u_m^-)_j + \lambda f(x,0)u_m^-\right] \end{aligned}$$

since $g(x,\xi) = 0$ if $\xi \le 0$. We conclude that

$$\|u_m^-\|_\ell^2 \le -J'(u_m)(u_m^-) + c\|u_m^-\|_\ell$$

whence $\{u_m^-\}$ is bounded. Next,

$$J'(u_m)(u_m^+) = \int_{\mathbb{R}^n} \sum a_{ij}(u_m^+)_i(u_m^+)_j - \lambda \int_{\mathbb{R}^n} \left(g(x,u_m)u_m^+ + f(x,0)u_m^+\right).$$

However,

$$K \ge J(u_m) = \frac{\|u_m\|_\ell^2}{2} - \lambda \int_{\mathbb{R}^n} [F_1(x,u_m) + f(x,0)u_m]$$

whence,

$$\frac{\|u_m^+\|_\ell^2}{2} - \lambda \int_{\mathbb{R}^n} F_1(x,u_m^+) \le K$$

since $\{u_m^-\}$ is bounded and $f(x,0) \le 0$. Hence

$$\frac{\|u_m^+\|_\ell^2}{2} - \lambda \int_{\mathbb{R}^n} \frac{g(x,u_m^+)u_m^+}{\mu} \le K$$

that is,

$$\frac{\|u_m^+\|_\ell^2}{2} + \frac{1}{\mu}\,[J'(u_m)(u_m^+) - \|u_m^+\|_\ell^2] + \frac{\lambda}{\mu}\int_{\mathbb{R}^n} f(x,0)u_m^+ \le K$$

and since $\mu > 2$,

$$\|u_m^+\|_\ell^2 - C_0\|u_m^+\|_\ell \le C_1[K + |J'(u_m)(u_m^+)|]$$

for some constants C_0, C_1. We conclude $\{u_m^+\}$ is bounded and therefore so is $\{u_m\}$.

Next,

$$\begin{aligned} J(u) &= \frac{\|u\|_\ell^2}{2} - \lambda\int_{\mathbb{R}^n} F_1(x,u) - \lambda\int_{\mathbb{R}^n} f(x,0)u \\ &\ge \tfrac{1}{2}\|u\|_\ell^2 - \frac{\lambda}{s+1}\int_{\mathbb{R}^n} f_1|u|^{s+1} - \lambda\int_{\mathbb{R}^n} f(x,0)u \\ &\ge \tfrac{1}{2}\|u\|_\ell^2 - \frac{\lambda}{s+1}C\|u\|_\ell^{s+1} - \lambda K\|u\|_\ell \\ &= \|u\|_\ell[\tfrac{1}{2}\|u\|_\ell - \frac{\lambda}{s+1}C\|u\|_\ell^s - \lambda K] \end{aligned}$$

for some constants C, K. Choosing λ small then shows that $J(u) \ge \alpha > 0$ for $\|u\|_\ell = 1$.

Finally, we recall $F_1(x,\alpha_0) > 0$ in Ω_0 and choose any function $0 < \omega \in C_0^\infty(\Omega_0)$. Consider

$$J(t\omega) = \frac{t^2\|\omega\|_\ell^2}{2} - \lambda\int_{\mathbb{R}^n} F_1(x,t\omega) - \lambda t\int_{\mathbb{R}^n} f(x,0)\omega.$$

As in [AY] and elsewhere, we observe that in $\text{supp}\,(\omega)$, $F_1(x,t) \geq t^\mu a_1 - a_2$ for some constants a_1, a_2, whence, since $\omega \in C_0^\infty$, we obtain:

$$\begin{aligned} J(t\omega) &\leq \frac{t^2\|\omega\|_\ell^2}{2} - \lambda \int_{\Omega_0} [(t\omega)^\mu a_1 - a_2] - \lambda t \int_{\mathbb{R}^n} f(x,0)\omega \\ &\leq \frac{t^2\|\omega\|_\ell^2}{2} - \lambda t^\mu \alpha_1 + \lambda(t+1)\alpha_2 \end{aligned}$$

for some constants α_1, α_2. We conclude immediately that $J(t\omega) < 0$ for all t large and the existence of a critical point of J – that is, of a solution of (1) – follows by the Mountain Pass Theorem for all small $\lambda > 0$.

Finally, for this solution we have:

$$\begin{aligned} \|u\|_\ell^2 &= \lambda \int_{\mathbb{R}^n} [g(x,u)u + f(x,0)u] \\ &\leq \lambda \int_{\mathbb{R}^n} [f_1|u|^{s+1} + f(x,0)u] \\ &\leq \lambda G[\|f_1\|_{L^{p_0}} \|u\|_\ell^{s+1} + \|f(x,0)\|_{L^2_{\sigma-1}} \|u\|_\ell]. \end{aligned}$$

From this inequality and the observation that $\|u\|_\ell$ must be bounded away from zero since $J(u)$ is, it immediately follows that

$$\|u\|_\ell \geq K\lambda^{-1/(s-1)}.$$

Also, let $\omega \in C_0^\infty$ be as above. We note:

$$\frac{dJ}{dt}(t\omega) = t\|\omega\|_\ell^2 - \lambda \int_{\mathbb{R}^n} g(x,t\omega)\omega - \lambda \int_{\mathbb{R}^n} f(x,0)\omega.$$

But since on $\text{supp}\,(\omega)$, $t^\mu a_1 - a_2 \le F_1(x,t) \le \frac{t}{\mu}\, g(x,t)$, we have, for some constants, α, β, γ and $t > 1$, that

$$\frac{dJ}{dt}\,(t\omega) \le t\alpha - \lambda t^{\mu-1}\beta + \lambda\gamma.$$

We recall that $\mu = s+1$. Hence if $t \ge C_1\lambda^{-1/(s-1)}$ then for λ small enough (depending on ω) $\frac{dJ}{dt} \le 0$. On the other hand, as shown earlier,

$$J(t\omega) \le \frac{t^2}{2}\,\|\omega\|_\ell^2 - \lambda t^\mu \alpha_1 + \lambda(t+1)\alpha_2,$$

that is,

$$J(t\omega) \le C\lambda^{-2/(s-1)}\,\frac{\|\omega\|_\ell^2}{2}\,.$$

We conclude $J(u) \le \lambda^{-2/(s-1)}K$ by the characterization of the Mountain Pass Critical Point. Now

$$J(u) = \frac{\|u\|_\ell^2}{2} - \lambda\int_{\mathbb{R}^n} F_1(x,u) - \lambda\int_{\mathbb{R}^n} f(x,0)u,$$

that is,

$$J(u) \ge \frac{\|u\|_\ell^2}{2} - \lambda\int_{\mathbb{R}^n} f(x,0)u - \frac{\|u\|_\ell^2}{\mu} + \lambda\int_{\mathbb{R}^n}\frac{f(x,0)u}{\mu}$$

i.e.,

$$C\lambda^{-2/(s-1)} \ge \|u\|_\ell^2 - \lambda\alpha\|u\|_\ell$$

and we obtain the estimate

$$\|u\|_\ell \leq H\lambda^{-1/(s-1)}.$$

We shall establish the positivity of the solution by means of the following lemmas. We assume henceforth that $f(x,0)f_1^\delta \in L^1$ with $\delta = (n-2)/[2n-(s+1)(n-2)]$, $g(x,t) \geq 0$ for $t > 0$, $f_1 > 0$, and meas $\{x|F_1(x,\alpha_0) = 0\} = 0$. Here meas A stands for the measure of A.

LEMMA 5.3. *For λ, $\varepsilon > 0$ sufficiently small, there exists a ball B and constant C, independent of u such that*

$$\text{meas}\left(\{x|u > \varepsilon f_1^\delta \lambda^{-1/(s-1)}\} \cap B\right) > C.$$

PROOF: Let $K = \{x|\ |u(x)| > \varepsilon f_1^\delta \lambda^{-1/s-1}\}$ and observe that $\delta(s+1) = p_0 - 1$. Without loss of generality, we may assume f_1 is locally bounded below by a positive constant. By Theorem 5.2 we have the estimate

$$C_0\lambda^{-2/(s-1)} \leq \|u\|_\ell^2 = \lambda\int_K [g(x,u)u + f(x,0)u] + \lambda\int_{\mathbb{R}^n-K}[g(x,u) + f(x,0)]u$$

$$\leq \lambda\int_K [f_1|u|^{s+1} + |f(x,0)|\ |u|]$$
$$+ \lambda\int_{\mathbb{R}^n-K}[f_1|u|^{s+1} + |f(x,0)|\ |u|]$$

$$\leq C\lambda[\|f_1\|_{L^{p_0}(K)}\|u\|_\ell^{s+1} + \|f(x,0)\|_{L^2_{\sigma-1}(K)}\|u\|_\ell]$$

$$+\varepsilon^{s+1}\lambda\int_{\mathbb{R}^n-K} f_1^{p_0}\lambda^{-\frac{s+1}{s-1}} + \varepsilon\int_{\mathbb{R}^n-K}|f(x,0)|f_1^{\delta}\lambda^{1-\frac{1}{s-1}}$$

$$\leq C_1[\|f_1\|_{L^{p_0}(K)}\lambda^{1-\frac{s+1}{s-1}} + \|f(x,0)\|_{L^2_{\varrho^{-1}}(K)}\lambda^{1-\frac{1}{s-1}}]$$
$$+\lambda^{1-\frac{s+1}{s-1}}\varepsilon^{s+1}\|f_1\|^{p_0}_{L^{p_0}(\mathbb{R}^n)} + \int_{\mathbb{R}^n}|f(x,0)|f_1^{\delta}\varepsilon\lambda^{1-\frac{1}{s-1}}$$

i.e.,

$$C_0 \leq C_1[\|f_1\|_{L^{p_0}(K)} + \|f(x,0)\|_{L^2_{\varrho^{-1}}(K)}\lambda^{s/(s-1)}]$$
$$+\varepsilon^{s+1}\|f_1\|^{p_0}_{L^{p_0}(\mathbb{R}^n)} + \varepsilon\lambda^{s/s-1}\|f(x,0)f_1^{\delta}\|_{L^1(\mathbb{R}^n)}.$$

Choosing, λ, ε sufficiently small, it follows that $\|f_1\|_{L^{p_0}(K)} \geq C_0/2$ with C_0 independent of u. Since $f_1 \in L^{p_0}(\mathbb{R}^n)$ by assumption, we conclude that there exists a ball $B \subset \mathbb{R}^n$ and constant C independent of u, λ and such that $\text{meas}\,(K \cap B) > C/4$, and the result follows.

Now let $0 < \omega \in E$ be such that

$$\ell\omega = -f(x,0) \triangleq -\widetilde{f}(x)$$

we then have

$$\ell(u + \lambda\omega) = \lambda g(x,u).$$

We do recall $g(x,t) = 0$ if $t \leq 0$, whence $u + \lambda\omega \geq 0$ and, in particular, $u^- \leq \lambda\omega$. It follows that for ε, λ small enough, $u > \varepsilon f_1^{\delta}\lambda^{-1/(s-1)}$

in $K \cap B$. Since $\mu F_1 \leq tg(x,t)$ implies $tg(x,t) \geq \mu F_1(x,\alpha_0)t^\mu \alpha_0^{-\mu}$ for $t > \alpha_0$, i.e. $g(x,t) \geq \mu t^s F_1(x,\alpha_0)\alpha_0^{-\mu}$ for x in $K \cap B$, $t > \alpha_0$. Therefore

$$\ell(u+\lambda\omega) \geq \lambda\alpha F_1(x,\alpha_0) f_1^{\delta s}\lambda^{-s/s-1}|_{K\cap B}$$

$$\geq \alpha F_1(x,\alpha_0) f_1^{\delta s}\lambda^{-1/s-1}|_{K\cap B}$$

for λ small enough and some $\alpha > 0$, since $f_1 > 0$ is bounded below in $K \cap B$.

Choose and fix any ball $B_1 \supset \overline{B}$. Note that $u + \lambda\omega \geq z$ where

$$\ell(z) = \alpha F_1(x,\alpha_0) f_1^{\delta s}\lambda^{-1/(s-1)}|_{K\cap B} \quad \text{in} \quad B_1$$

$$z = 0 \quad \text{on} \quad \partial B_1$$

and hence

$$\inf_{\partial B}(u+\lambda\omega) \to +\infty \quad \text{as} \quad \lambda \to 0,$$

since $F_1(x,\alpha_0)f_1^{\delta s}$ is nontrivial in K. Indeed, observe that if G_1 denotes the Green's function for B_1 then $G_1(x,y) > \pi > 0$ for some $\pi > 0$ and any $(x,y) \in \overline{B}\times\overline{B}$, [St]. It follows that $z(x) \geq \pi\alpha\lambda^{-1/(s-1)}\|F_1(\cdot,\alpha_0)f_1^{\delta s}\|_{L^1(K\cap B)}$ and we obtain the result.

LEMMA 5.4. *There exists a function $C(\lambda)$ such that for $|x|$ near infinity:*

$$(u+\lambda\omega)(x) \geq C(\lambda)\,\frac{1}{|x|^{n-2}}.$$

and $C(\lambda) \to \infty$ *as* $\lambda \to 0$.

PROOF: Suppose first that $\ell = -\Delta$. Then this is immediate from the three sphere theorem, since $\ell(u + \lambda\omega) \geq 0$, $u + \lambda\omega \geq 0$. To deal with the more general case, consider a nested sequence of balls $\{B_i\}$ exhausting $\mathbb{R}^n$. Let G_i, K_i denote the Green's functions associated with ℓ and $-\Delta$ respectively with Dirichlet conditions on ∂B_i. Select a point x and choose a ball B_0 centered at x. Note that $CK_i(x,y) \leq G_i(x,y) \leq DK_i(x,y)$ for any $y \in \partial B_0$ and i large enough with C, D constants independent of i, x, y, and the radius of B_0, [St]. It follows that $u(x) + \lambda\omega(x) \geq C\tau_i(x)$ where $-\Delta\tau_i = \alpha F_1(x,\alpha_0) f_1^{\delta s} \lambda^{-1/s-1}|_{K\cap B}$, $\tau_i = 0$ on ∂B_i. Passing to the limit as $i \to \infty$ gives the estimate.

To conclude the preliminary steps, we need only show that $\omega \leq \frac{d}{|x|^{n-2}}$ if $\ell\omega = -\widetilde{f}(x)$ for some d. Recall $\widetilde{f}(x) = f(x,0)$ and that the arguments in Lemma 5.4 indicate that it suffices to show this for $\ell = -\Delta$. To see this, choose and fix a point x_0, and denote by ω_i the solution to $\ell(\omega_i) = -\widetilde{f}|_{B_{\varepsilon i}(x_0)}$, $\omega_i = 0$ on $\partial B_i(x_0)$ for some small fixed ε, and with $B_r(x_0)$ denoting the sphere of radius r centered at x_0. We then note that $\omega_i \uparrow \omega$ pointwise and $\omega_i \to \omega$ weakly in E, as $i \to \infty$. Let v_i, v represent the analogous functions to ω_i, ω with ℓ replaced by $-\Delta$. Again from the results of [St], [GT], we have $\omega_i(x_0) \leq Dv_i(x_0)$, with D independent of x_0, i. Passing to the limit, we obtain $\omega(x_0) \leq Dv(x_0)$.

Consequently assume $-\Delta\omega = -\widetilde{f}(x)$. We immediately obtain from the Riesz Potential formula that

$$\omega(x) = K_1 \int_0^\infty \frac{1}{t^{n-1}} \|\widetilde{f}\|_{L^1(B_t(x))} dt$$

and thus require that

$$(*) \qquad |x|^{n-2} \int_0^\infty \frac{1}{t^{n-1}} \|\widetilde{f}\|_{L^1(B_t(x))} \leq C$$

for $|x|$ large. It is easy to give sufficient conditions on $\widetilde{f}(x)$ for this to be the case. As a specific example we have:

LEMMA 5.5. *Assume* $\widetilde{f} \in L^1$ *and* $|x|^{n/q} \|\widetilde{f}\|_{L^p(\mathbb{R}^n - B_{|x|}(0))} \leq C$ *with* $p > \frac{n}{2}$, $\frac{1}{q} + \frac{1}{p} = 1$, *then* $(*)$ *holds.*

PROOF: We may write

$$\begin{aligned} \int_0^\infty \frac{1}{t^{n-1}} \|\widetilde{f}\|_{L^1(B_t(x))} dt &= \int_{|x|/2}^\infty \frac{1}{t^{n-1}} \|\widetilde{f}\|_{L^1(B_t(x))} + \int_0^{|x|/2} \frac{1}{t^{n-1}} \|\widetilde{f}\|_{L^1(B_t(x))} \\ &= J_1 + J_2. \end{aligned}$$

Now $J_1 \leq C\|\widetilde{f}\|_{L^1(\mathbb{R}^n)} |x|^{2-n}$ while

$$\begin{aligned} J_2 &= \int_0^{|x|/2} \frac{1}{t^{n-1}} \|\widetilde{f}\|_{L^1(B_t(x))} \\ &\leq C \int_0^{|x|/2} t^{1-n+\frac{n}{q}} \|\widetilde{f}\|_{L^p(B_t(x))} \\ &\leq C\|\widetilde{f}\|_{L^p(B_{|x|/2}(x))} \left(\frac{|x|}{2}\right)^{2-n+\frac{n}{q}} \\ &\leq C\|\widetilde{f}\|_{L^p(\mathbb{R}^n - B_{|x|/2}(0))} \left[\frac{|x|}{2}\right]^{n/q} \cdot |x|^{2-n} \end{aligned}$$

and the estimate follows.

THEOREM 5.6. *Assume* $f_1 > 0$, $f(x,0)f_1^{\delta} \in L^1$ *with* $\delta = \frac{(n-2)}{[2n-(s+1)(n-2)]}$, $g(x,t) \geq 0$ *if* $t > 0$, *meas* $\{x|F_1(x,\alpha_0) = 0\} = 0$. *Further assume that* $f(x,0) \equiv \widetilde{f}(x)$ *satisfies the conditions of Lemma 5.5. Then the solution* u *found in Theorem 5.2 is positive.*

The Proof of Theorem 5.6 now follows immediatly from Lemmas 5.1, 5.3, 5.4 and 5.5.

5.4. Existence Result for (5.1.2)

In this section, we extend the existence result obtained in Section 5.3 for the scalar equation (5.1.1) to system (5.1.2). Our basic method is similar to that used in Section 5.3. We start with the following initial assumptions on f^k :

(AI) $f^k \in C^{\alpha}_{loc}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $\alpha \in (0,1)$, $k = 1, \ldots, N$.

(AII) $f^k(x,\vec{0}) \leq 0$, $f^k(x,\vec{0}) \not\equiv 0$, $f^k(x,\vec{\xi}) = f^k(x,\vec{\xi}^+)$ for $k = 1, \ldots, N$.

(AIII) $f^k(x,\vec{t})$ satisfies assumption (A3) of Chapter 4.

(AIV) $|\vec{f}(x,\vec{t})| \leq \widehat{f}_0(x) + \widehat{f}_1(x)|\vec{t}|^s$, $x \in \mathbb{R}^n$, $\vec{t} \in \mathbb{R}^N$,

with $\widehat{f}_0 \in L^2_{\sigma^{-1}} \cap L^{\infty}$, $0 < \widehat{f}_1 \in L^{p_0} \cap L^{\infty}(\mathbb{R}^n)$; where s, p_0 and $\vec{f}(x,\vec{t})$ are as in Chapter 4. $L^2_{\sigma^{-1}}$ is as defined in Section 5.2.

(AV) $f^k(x,\vec{0}) \in L^2_{\sigma-1} \cap L^\infty$, $f^k(x,\vec{0})\widehat{f_1^\delta} \in L^1(\mathbb{R}^n)$ for $k = 1,\dots,N$ and $\delta = \frac{n-2}{2n-(s+1)(n-2)}$.

We observe, as in Chapter 4, that in view of assumption (AIII), there exists a differentiable function $F(x,\vec{u})$ such that

$$\nabla_{\vec{u}} F(x,\vec{u}) = \vec{f}(x,\vec{u}).$$

If we let $\widetilde{G}(\vec{u}) = \int_{\mathbb{R}^n} F(x,\vec{u})dx$ and $\widetilde{J}(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2_\ell - \lambda\widetilde{G}(\vec{u})$ for $\vec{u} \in E$, we have the following lemma:

LEMMA 5.7. *(a)* $\widetilde{G}$ *and* $\widetilde{J}$ *are weakly lower semicontinuous with*

$$\widetilde{G}'(\vec{u})(\vec{\varphi}) = \int_{\mathbb{R}^n} \vec{f}(x,\vec{u}) \cdot \vec{\varphi}\, dx.$$

(b) $\widetilde{G}'$ *is continuous and compact from* E *to* E.

As before, we observe that by virtue of Lemma 5.7, the weak solutions $\vec{u}$ of (5.1.2) in E, which may be described as the critical points of $\widetilde{J}$, satisfy:

LEMMA 5.8.

(a) $\vec{u} \in L^p(\mathbb{R}^n)$, $\frac{2n}{n-2} \le p \le \infty$

(b) $\lim_{|x|\to\infty} u^k = 0$ *for* $k = 1,\dots,N$.

The proofs of Lemmas 5.7 and 5.8 essentially follow as in Chapter 4.

In view of (AII), we may write

$$f^k(x,\vec{u}^+) =: g^k(x,\vec{u}) + f^k(x,\vec{0}) \tag{5.4.1}$$

with $g^k(x,\vec{0}) = 0$ for all k. We make the following additional assumptions on $g^k(x,\vec{u})$:

(BI) $g^k(x,\vec{u}) \geq 0$ for $x \in \mathbb{R}^n$, $\vec{u} \geq \vec{0}$; with $g^k(x,\vec{u}) > 0$ if every $u^j > 0$, $j = 1,\ldots,N$.

(BII) $\frac{1}{\mu}\,[\sum_{k=1}^{N} g^k(x,\vec{t})t^k] \geq \widetilde{F}_1(x,\vec{t})$ for all $x \in \mathbb{R}^n$, $\vec{t} \geq 0$, where $\mu > 2$ and $\widetilde{F}_1$ denotes the potential associated with $\vec{g}$.

(BIII) $g^k(x,\vec{t}) = 0$ whenever $t^k = 0$ and $t^j \geq 0$ for $j \neq k$, $x \in \mathbb{R}^n$; and

$$g^k(x,\vec{t}\,) \geq \alpha(x)(t^k)^s \quad \text{for some} \quad \alpha > 0,$$

whenever $t^k > 0$ and $t^j \geq 0$ for $j \neq k$, and $x \in \mathbb{R}^n$.

We observe that $|\vec{g}(x,\vec{t}\,)| \leq \widehat{f}_1(x)|\vec{t}\,|^s$ and $g^k(x,u^1,\ldots,u^j,\ldots,u^N) = g^k(x,u^1,\ldots,0,\ldots,u^N)$ if $u^j < 0$ and $u^i \geq 0$ for $j \neq i$, $k = 1,\ldots,N$.

From (5.4.1) and assumptions (AII) and (BIII), we see that $\vec{f}(x,\vec{u})$ is quasi-irreducible in the sense that if $\vec{u}$ is a solution, then no component can be identically zero. We now have:

THEOREM 5.9. *Under the assumptions (AI) - (AV) and (BI) - (BIII), system (5.1.2) has a solution $\vec{u} \in E$ for λ small and*

$$M\lambda^{-\frac{1}{s-1}} \leq \|\vec{u}\|_E \leq H\lambda^{-\frac{1}{s-1}}$$

for some M, H independent of $\vec{u}$.

PROOF: The proof essentially follows as in the scalar case; we present it only for completeness. We show first that the $P-S$ condition is satisfied. Assume $\{\vec{u}_m\}$ is a sequence with $\{\widetilde{J}(\vec{u}_m)\}$ bounded and $\widetilde{J}'(\vec{u}_m) \to 0$. Following a similar argument as in Chapter 4, and in view of Lemma 5.7, it suffices to show that $\|\vec{u}\|_\ell$ is bounded. We set $\vec{u}^- = ((u^1)^-, (u^2)^-, \ldots, (u^N)^-)$. Since $\widetilde{J}'(\vec{u}_m) \to 0$, we have for m large

$$\|\vec{u}_m^-\|_\ell^2 - \lambda \sum_{k=1}^{N} \left[\int_{\mathbb{R}^n} g^k(x, \vec{u}_m)(u_m^k)^- + f^k(x, \vec{0})(u_m^k)^- \right] \leq \|\vec{u}_m^-\|_\ell$$

which implies

$$\|\vec{u}_m^-\|_\ell^2 \leq \lambda \sum_{k=1}^{N} \left(f^k(x, \vec{0})(u_m^k)^- \right) + \|\vec{u}_m^-\|_\ell$$

since $g^k(x, \vec{t}) \equiv 0$ if $\vec{t} \leq \vec{0}$.

We conclude that

$$\|\vec{u}_m^-\|_\ell^2 \leq C\|\vec{u}_m^-\|_\ell + \|\vec{u}_m^-\|_\ell$$

from where it follows that $\{\vec{u}_m^-\}$ is bounded.

Next

$$\widetilde{J}'(\vec{u}_m)(\vec{u}_m^+) = \sum_{k=1}^{N} \Big[\int_{\mathbb{R}^n} \sum a_{ij} (u_m^k)_\ell^+ (u_m^k)_j^+ - \lambda \Big(\int_{\mathbb{R}^n} g^k(x, \vec{u}_m)(u_m^k)^+ \int_{\mathbb{R}^n} f^k(x, \vec{0})(u_m^k)^+ \Big) \Big]. \tag{5.4.2}$$

However, by the boundedness assumption on $\{\widetilde{J}(\vec{u}_m)\}$, we have

$$K \geq \widetilde{J}(\vec{u}_m)$$
$$= \frac{\|\vec{u}_m\|_\ell^2}{2} - \lambda \int_{\mathbb{R}^n} \widetilde{F}_1(x, \vec{u}_m) - \lambda \sum_{k=1}^{N} \int_{\mathbb{R}^n} f^k(x, \vec{0}) u_m^k,$$

whence,

$$\frac{\|\vec{u}_m^+\|_\ell^2}{2} - \lambda \int_{\mathbb{R}^n} \widetilde{F}_1(x, \vec{u}_m^+) \leq K$$

since $\{\vec{u}_m^-\}$ is bounded and $f^k(x, \vec{0}) \leq 0$. Hence

$$\frac{\|\vec{u}_m^+\|_\ell^2}{2} - \lambda \sum_{k=1}^{N} \int_{\mathbb{R}^n} \frac{g^k(x, \vec{u}_m^+)}{\mu} (u_m^k)^+ \leq K$$

by (BII); that is,

$$\frac{\|\vec{u}_m^+\|_\ell^2}{2} + \frac{1}{\mu} \left[\widetilde{J}'(\vec{u}_m)(\vec{u}_m^+) - \|\vec{u}_m^+\|_\ell^2\right]$$
$$+ \frac{\lambda}{\mu} \sum_{k=1}^{N} \int_{\mathbb{R}^n} f^k(x, \vec{0})(u_m^k)^+ \leq K$$

by using (5.4.2); and since $\mu > 2$,

$$\|\vec{u}_m^+\|_\ell^2 - C_0 \|\vec{u}_m^+\|_\ell \leq C_1 [K + \widetilde{J}'(\vec{u}_m)(\vec{u}_m^+)]$$

for some constants C_0, C_1. We conclude that $\{\vec{u}_m^+\}$ is bounded and therefore so is $\{\vec{u}_m\}$.

Next,

$$\begin{aligned}\widetilde{J}(\vec{u}) &= \frac{\|\vec{u}\|_\ell^2}{2} - \lambda \int_{\mathbb{R}^n} \widetilde{F}_1(x,\vec{u}) - \lambda \sum_{k=1}^N \int_{\mathbb{R}^n} f^k(x,\vec{0})u^k \\ &\geq \frac{1}{2}\|\vec{u}\|_\ell^2 - \lambda C_1 \int_{\mathbb{R}^n} \widehat{f}_1 |\vec{u}|^{s+1} - \lambda C_2 \|\vec{u}\|_\ell \\ &\geq \frac{1}{2}\|\vec{u}\|_\ell^2 - \lambda C \|\vec{u}\|_\ell^{s+1} - \lambda C_2 \|\vec{u}\|_\ell \\ &= \|\vec{u}\|_\ell [\frac{1}{2} - \lambda C \|\vec{u}\|_\ell^{s-1} - \lambda C_2]\end{aligned}$$

for some constants C, C_2. Choosing first of all λ small and then $\rho = \|\vec{u}\|_\ell$ small, we have $J(\vec{u}) \geq \alpha > 0$ for $\|\vec{u}\|_\ell = \rho$.

Let us now choose any function $\vec{0} < \vec{\omega} \in [C_0^\infty(\mathbb{R}^n)]^N$ and consider

$$\widetilde{J}(t\vec{\omega}) = \frac{t^2}{2}\|\vec{\omega}\|_\ell^2 - \lambda \int_{\mathbb{R}^n} \widetilde{F}_1(x,t\vec{\omega}) - \lambda t \sum_{k=1}^N \int_{\mathbb{R}^n} f^k(x,\vec{0})\omega^k.$$

We observe that $\widetilde{F}_1(x,t\vec{\omega}) \geq \widetilde{F}_1(x,\vec{\omega})t^\mu - a_2$ for some constant a_2; hence we have

$$\widetilde{J}(t\vec{\omega}) \leq \frac{t^2}{2}\|\vec{\omega}\|_\ell^2 - \lambda t^\mu a_1 + \lambda t a_2$$

for some constants α_1, α_2. We thus conclude that $\widetilde{J}(t\vec{\omega}) < 0$ for all t large and the existence of a critical point of $\widetilde{J}$ – that is, existence of a solution of (5.1.2) follows by the Mountain Pass Theorem.

Finally, for this solution we have:

$$\begin{aligned}\|\vec{u}\|_\ell^2 &= \lambda \sum_{k=1}^{N} [\int_{\mathbb{R}^n} g^k(x,\vec{u})(u^k) + \int_{\mathbb{R}^n} f^k(x,\vec{0})u^k] \\ &\leq \lambda C_1 \int_{\mathbb{R}^n} \widehat{f}_1 |\vec{u}|^{s+1} + \lambda \sum_{k=1}^{N} \int_{\mathbb{R}^n} f^k(x,\vec{0})u^k \\ &\leq \lambda C[\|\widehat{f}_1\|_{L^{P_0}} \|\vec{u}\|_\ell^{s+1} + \|f^k(x,\vec{0})\|_{L^2_{\sigma-1}} \|\vec{u}\|_\ell].\end{aligned}$$

From this inequality, it follows that

$$\|\vec{u}\|_\ell \geq M\lambda^{-\frac{1}{s-1}}.$$

Now, let $\vec{\omega}$ be as above. We observe that

$$\frac{d\widetilde{J}}{dt}(t\vec{\omega}) = t\|\vec{\omega}\|_\ell^2 - \lambda \sum_{k=1}^{N} [\int_{\mathbb{R}^n} g^k(x,t\vec{\omega})\omega^k - \int_{\mathbb{R}^n} f^k(x,\vec{0})\omega^k].$$

But $\sum_{k=1}^{N} (t^k)^\mu a_1 - a_2 \leq \widetilde{F}_1(x,\vec{t}) \leq \sum \frac{t^k}{\mu} g^k(x,\vec{t})$, therefore

$$\frac{d\widetilde{J}}{dt}(t\vec{\omega}) \leq t\alpha - \lambda t^\mu \beta + \lambda\gamma.$$

We may set $\mu = s+1$. Hence if $t \geq \lambda^{-\frac{1}{s-1}}$, then for λ small enough (depending on $\vec{\omega}$), $\frac{d\widetilde{J}}{dt} \leq 0$.

On the other hand,

$$\widetilde{J}(t\vec{\omega}) \leq \frac{t^2}{2} \|\vec{\omega}\|_\ell^2 - \lambda t^\mu \alpha_1 + \lambda t \alpha_2$$

that is,

$$\widetilde{J}(t\vec{\omega}) \leq C \frac{\|\vec{\omega}\|_\ell^2}{2} \lambda^{\frac{-2}{s-1}}.$$

We conclude that $\widetilde{J}(\vec{u}) \leq K\lambda^{\frac{-2}{\rho-1}}$ by the characterization of the Mountain Pass critical point. Now

$$\widetilde{J}(\vec{u}) = \frac{1}{2}\,\|\vec{u}\|_\ell^2 - \lambda \int_{\mathbb{R}^n} \widetilde{F}_1(x,\vec{u}) - \lambda \sum_{k=1}^{N} \int_{\mathbb{R}^n} f^k(x,\vec{0})u^k.$$

But

$$\frac{1}{\mu}\,\|\vec{u}\|_\ell^2 = \lambda \sum_{k=1}^{N} [\int_{\mathbb{R}^n} \frac{g^k(x,\vec{u})u^k}{\mu} + \int_{\mathbb{R}^n} \frac{f^k(x,\vec{0})u^k}{\mu}]$$

that is,

$$\widetilde{J}(\vec{u}) \geq \frac{1}{2}\,\|\vec{u}\|_\ell^2 - \lambda \sum_{k=1}^{N} \int_{\mathbb{R}^n} f^k(x,\vec{0})u^k - \frac{1}{\mu}\,\|\vec{u}\|_\ell^2 + \lambda \sum_{k=1}^{N} \int_{\mathbb{R}^n} \frac{f^k(x,\vec{0})u^k}{\mu}$$

that is,

$$K\lambda^{-\frac{2}{\rho-1}} \geq \|\vec{u}\|_\ell^2 - \lambda\alpha\|\vec{u}\|_\ell$$

and the estimate

$$\|\vec{u}\|_\ell \leq H\lambda^{-\frac{1}{\rho-1}}$$

follows.

THEOREM 5.10. *In addition to the hypotheses of Theorem 5.9, assume* $f^k(x,\vec{0}) \in L^1$ *and*

$$|x|^{n/q}\|f^k(x,\vec{0})\|_{L^p(\mathbb{R}^n \setminus B_{|x|}(0))} \leq C$$

with $p > \frac{n}{2}$, $\frac{1}{q} + \frac{1}{p} = 1$, *then the solution* $\vec{u}$ *found in Theorem 5.9 is such that* $\sum_{k=1}^{N} u^k > 0$.

We prove Theorem 5.10 by a series of lemmas.

LEMMA 5.11. *For* $\varepsilon > 0$, *there exists a ball* B, *independent of* $\vec{u}$, *such that*

$$meas\,(\{x||\vec{u}| > \varepsilon \widehat{f}_1^{\delta} \lambda^{-\frac{1}{s-1}}\} \cap B) > C > 0.$$

Recall $meas\,(S)$ *is the measure of the set* S.

PROOF: Let $\widetilde{K} = \{x|\ |\vec{u}| > \varepsilon \widehat{f}_1^{\delta} \lambda^{-\frac{1}{s-1}}\}$. By Theorem 5.9, we have the estimate

$$\begin{aligned}
C_0 \lambda^{\frac{-2}{s-1}} &\le \|\vec{u}\|_{\ell}^2 \\
&= \lambda \sum_{k=1}^{N} \int_{\widetilde{K}} g^k(x,\vec{u}) u^k + \lambda \sum_{k=1}^{N} \int_{\widetilde{K}} f^k(x,\vec{0}) u^k \\
&\quad + \lambda \sum_{k=1}^{N} [\int_{\mathbb{R}^n \setminus \widetilde{K}} \left(g^k(x,\vec{u}) u^k + |f^k(x,\vec{0})| u^k\right)] \\
&\le \lambda C_1 \int_{\widetilde{K}} \widehat{f}_1 |\vec{u}|^{s+1} + \lambda \sum_{k=1}^{N} \int_{\widetilde{K}} |f^k(x,\vec{0})|\ |u^k| \\
&\quad + \lambda C_1 \int_{\mathbb{R}^n \setminus \widetilde{K}} [\widehat{f}_1 |\vec{u}|^{s+1}] + \lambda \sum_{k=1}^{N} \int_{\mathbb{R}^n \setminus \widetilde{K}} |f^k(x,\vec{0})|\ |u^k| \\
&\le \lambda C_2 [\|\widehat{f}_1\|_{L^{p_0}(\widetilde{K})} \|\vec{u}\|_{\ell}^{s+1} + \sum_{k=1}^{N} \|f^k(x,\vec{0})\|_{L^{\sigma^{-1}}(\widetilde{K})} \|\vec{u}\|_{\ell}]
\end{aligned}$$

$$+\varepsilon\lambda C_3\int_{\mathbb{R}^n\setminus\widetilde{K}}\widehat{f}_1^{p_0}\lambda^{-\frac{\sigma+1}{\sigma-1}}+\varepsilon C_3\sum_{k=1}^{N}\int_{\mathbb{R}^n\setminus\widetilde{K}}|f^k(x,\vec{0})|\widehat{f}_1^{\delta}\lambda^{1-\frac{1}{\sigma-1}}$$

$$\leq C_4[\|\widehat{f}_1\|_{L^{p_0}(\widetilde{K})}\lambda^{1-\frac{(\sigma+1)}{\sigma-1}}+\sum_{k=1}^{N}\|f^k(x,\vec{0})\|_{L_{\sigma^{-1}}(\widetilde{K})}\lambda^{1-\frac{1}{\sigma-1}}]$$

$$+\lambda^{1-\frac{(\sigma+1)}{\sigma-1}}C_3\varepsilon\|\widehat{f}_1\|^{p_0}_{L^{p_0}(\mathbb{R}^n)}+C_3\sum_{k=1}^{N}\int_{\mathbb{R}^n}|f^k(x,\vec{0})|\,\widehat{f}_1^{\delta}\varepsilon\lambda^{1-\frac{1}{\sigma-1}}$$

that is,

$$C_0\leq C[\|\widehat{f}_1\|_{L^{p_0}(\widetilde{K})}+\sum_{k=1}^{N}\|f^k(x,\vec{0})\|_{L_{\sigma^{-1}}(\widetilde{K})}\lambda^{\frac{\sigma}{\sigma-1}}$$

$$+\varepsilon\|\widehat{f}_1\|^{p_0}_{L^{p_0}(\mathbb{R}^n)}+\varepsilon\lambda^{\frac{\sigma}{\sigma-1}}\sum_{k=1}^{N}\|f^k(x,\vec{0})\widehat{f}_1^{\delta}\|_{L^1(\mathbb{R}^n)}].$$

Choosing λ, ε sufficiently small, we conclude that $\|\widehat{f}_1\|_{L^{p_0}(\widetilde{K})}\geq\frac{C_0}{2}$ with C_0 independent of $\vec{u}$. Since $\widehat{f}_1\in L^{p_0}(\mathbb{R}^n)$ by assumption, we conclude that there exists a ball $B\subset\mathbb{R}^n$ independent of $\vec{u}$, λ and such that meas $(\widetilde{K}\cap B)>\frac{C}{4}$.

We observe that

$$\text{meas}\,(\widetilde{K}\cap B)\leq\sum_{k}\text{meas}\,(\{x|\,|u^k|>\varepsilon_1\widehat{f}_1^{\delta}\lambda^{-\frac{1}{\sigma-1}}\}\cap B);$$

for some ε_1 independent of $\vec{u}$, and by Lemma 5.11 we conclude that there is at least one k, denoted by k^*, such that

$$\text{meas}\,(\{x|\,|u^{k^*}|>\varepsilon_1\widehat{f}_1^{\delta}\lambda^{-\frac{1}{\sigma-1}})\cap B)>C_1>0.$$

We let $\widetilde{K}_*=\{x|\,|u^{k^*}|>\varepsilon_1\widehat{f}_1^{\delta}\lambda^{-\frac{1}{\sigma-1}}]$.

Now let $\vec{0} < \vec{\omega} \in E$ be such that

$$\ell\omega^k = -f^k(x, \vec{0}).$$

We then have

$$\ell(\sum u^k + \lambda \sum \omega^k) = \lambda \sum g^k(x, \vec{u}), \quad k = 1, 2, \ldots, N.$$

Recall that $g^k(x, \vec{t}) \equiv 0$ if $\vec{t} \leq \vec{0}$, for each k, whence

$$\sum u^k + \lambda \sum \omega^k \geq 0 \quad \text{and, in particular,} \quad \sum (u^k)^- \leq \lambda \sum \omega^k.$$

It follows that for λ small enough, $u^{k^*} > \varepsilon \widehat{f}_1^{\delta} \lambda^{-\frac{1}{s-1}}$ in $\widetilde{K}_* \cap B$. By (BIII), $g^{k^*}(x, \vec{t}) \geq \alpha(x)(t^{k^*})^s$ with $t^{k^*} > 0$, $\alpha > 0$ in $\widetilde{K}_* \cap B$ in particular.

Therefore,

$$\begin{aligned} \ell(\sum u^k + \lambda \sum \omega^k) &\geq \lambda \alpha (u^{k^*})^s|_{\widetilde{K}_*} \\ &\geq \lambda \alpha \varepsilon_1 \widehat{f}_1^{\delta s} \lambda^{-\frac{s}{s-1}}|_{\widetilde{K}_*} \\ &\geq \alpha_1 \widehat{f}_1^{\delta s} \lambda^{-\frac{1}{s-1}}|_{\widetilde{K}_*}. \end{aligned}$$

Choose and fix any ball $B_1 \supset B$. We note that $\sum u^k + \lambda \sum \omega^k \geq z$ where

$$\ell z = \alpha_1 \widehat{f}_1^{\delta s} \lambda^{-\frac{1}{s-1}}|_{\widetilde{K}_*} \quad z = 0 \quad \text{on} \quad \partial B_1$$

and hence

$$\inf_{\partial B} (\sum u^k + \lambda \sum \omega^k) \to +\infty \quad \text{as} \quad \lambda \to 0.$$

By the assumption that $g^k(x,\vec{t}) \geq 0$ if $\vec{t} \geq \vec{0}$, for each k, it follows that

$$\ell(\sum u^k + \lambda \sum \omega^k) \geq 0 \quad \text{in} \quad \mathbb{R}^n \backslash B.$$

LEMMA 5.12. *There exists a constant* C *such that near infinity,*

$$(\sum u^k + \lambda \sum \omega^k) \geq C \inf_{x \in \partial B_1} (\sum u^k + \lambda \sum \omega^k) \cdot \frac{1}{|x|^{n-2}}.$$

PROOF: Essentially the same as for the scalar case.

To conclude the steps, we need only show that $\sum \omega^k \leq \frac{d}{|x|^{n-2}}$ if $\ell \omega^k = -f^k(x,\vec{0})$, for some d. As observed in Section 5.3, it suffices to show this for $\ell = -\Delta$. Consequently assume $-\Delta \omega^k = -f^k(x,\vec{0})$. We immediately obtain that

$$\omega^k(x) = C_1 \int_1^\infty \frac{1}{t^{n-1}} \|f^k(x,\vec{0})\|_{L^1(B_t(x))} dt$$

and thus require that

$$(*) \qquad |x|^{n-2} \int_0^\infty \frac{1}{t^{n-1}} \|f^k(x,\vec{0})\|_{L^1(B_t(x))} \leq C$$

for $|x|$ large. But $(*)$ holds, as shown in Section 5.3; and the proof of Theorem 5.10 now follows from lemmas 5.11 and 5.12.

5.5. Examples

To illustrate the above results, and motivated by the positone cases considered in [AY], we examine:

$$\text{(5.5.1)} \qquad \begin{cases} -\Delta u = \lambda[g(x)u^s - h(x)] & \text{in } \mathbb{R}^n \\ \lim\limits_{|x|\to\infty} u = 0 \end{cases}$$

with $g,\ h > 0,$ smooth, bounded $n > 2$ and $1 < s < \frac{n+2}{n-2}$.

COROLLARY 5.7. *Assume* $g \in L^{p_0}$ *and* $|x|h \in L^2,\ h \in L^1$. *Furthermore assume* $|x|^{n/q}\|h\|_{L^p(R^n - B_{|x|}(0))} \le C$ *for some* $p > n/2,$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then (5.4.1) has a positive decaying solution for* λ *small.*

PROOF: We need only observe that $h \in L^1 \cap L^\infty$ implies that $h \in L^p$ for any $p,$ and in particular we conclude $hg^\delta \in L^1$ by Hölder's Inequality, as $\delta < p_0$. Finally note that $|x|h \in L^2$ and $h \in L^1$ imply $h \in L^2_{\sigma-1}$ and all the conditions of Theorem 5.6 hold.

If we further assume g, h are radial functions, i.e., $g = g(|x|),$ $h = h(|x|)$ then the conditions of Corollary 5.7 hold immediately if $h(|x|) = 0(|x|^{-b}),$ and $g(|x|) = O(|x|^{-a})$ at infinity, for $b > n$ and $[2n - (s+1)(n-2)] < 2a$ or, equivalently,

$$\max\{(n+2-2a)/(n-2), 1\} < s < (n+2)/(n-2).$$

Consider now the system

$$(5.5.2)\qquad \begin{cases} -\Delta u &= \lambda\{g(x)[(u^+)^\sigma + \alpha_3(u^+)^{\alpha_1}(v^+)^{\alpha_2}] - h^1(x)\} \\ -\Delta v &= \lambda\{g(x)[(v^+)^\sigma + \alpha_2(u^+)^{\alpha_3}(v^+)^{\alpha_4}] - h^2(x)\} \end{cases}$$

in $\mathbb{R}^n$ with $g,\ h^1,\ h^2 > 0,$ smooth, bounded and $n > 2.$ Assume $\alpha_i \geq 0$ for $i = 1,\ldots,4,$ $\alpha_4 + 1 = \alpha_2,$ $\alpha_1 + 1 = \alpha_3,$ $1 < \alpha_1 + \alpha_2 \leq \sigma < \frac{n+2}{n-2}$. If g is as in the example above and $h^1,\ h^2$ satisfy the same conditions as the above $h,$ then (5.5.2) has a decaying solution (u,v) with $u+v$ positive (and at least one component positive) for λ small.

CHAPTER 6

CONCLUSIONS

In this thesis we have studied, semilinear elliptic systems in the entire space $\mathbb{R}^n$ under three basic problem types:

(a) nonvariational problems;

(b) variational problems; and

(c) nonpositone problems.

Various existence results were established for these problems. We were primarily interested in positive decaying (to zero) solutions and we concentrated on the case of superlinear and subcritical nonlinearities. For the existence proofs, we used topological degree arguments and critical point theory as appropriate. The critical point theory was employed via the Mountain Pass theorem of Ambrosetti and Rabinowitz. We used various weighted spaces E which were locally isomorphic to the usual Sobolev space $W_0^{1,2}$. The use of these spaces enabled us to deal with some of the difficulties which arise when working in the unbounded domain $\mathbb{R}^n$ – such as the lack of compactness and the difficulty of guaranteeing the right behaviour at infinity of the solutions. The componentwise positivity of our solutions followed from the assumption of quasi-irreducibility on $\vec{f}(x, \vec{u})$.

In this thesis, we actually obtained existence results for nonvariational semilinear elliptic systems with superlinear and subcritical nonlinearities; we

improved upon the results of Chaljub-Simon and Volkmann [CV] for scalar equations, and extended those of Noussair and Swanson [NS3] for systems with variational structure. Our results on the nonpositone problems in the entire space $\mathbb{R}^n$ are, to the best of our knowledge, the first in this direction, as are also the criteria for positive solutions of reducible problems.

We point out that the use of topological degree arguments to obtain some of our existence results depended on the existence of global a priori estimates for all positive solutions of the system under consideration. Such global estimates were obtained in this thesis by imposing certain monotonicity conditions on the nonlinear terms $f^i(x,\vec{u})$, $1 \leq i \leq N$, in the x variable – in order to be able to cover the "full" subcritical growth range, $1 < \beta < \frac{n+2}{n-2}$. These monotonicity conditions also imposed restrictions on the elliptic operators that can be considered – for instance, operators that did not contain first order terms. In the chapter where we considered operators admitting first order terms, we were only able to achieve a growth of order $|\vec{u}|^\beta$, $1 < \beta < \frac{n}{n-2}$. It would certainly be interesting to extend our results to the case of nonvariational elliptic systems with general elliptic operators, without the monotonicity conditions in the x variable of the nonlinear terms, and yet cover the "full" growth range. For the nonpositone problems, we only considered the case of a variational system and obtained existence results for small values of the parameter λ. It would equally be interesting to extend these results to nonvariational systems, to be able to achieve componentwise positivity and to investigate the

existence/nonexistence question for λ large. Finally, we mention that we have not considered in this thesis elliptic systems with singular nonlinearities. We plan to investigate these outstanding problems in the very near future.

BIBLIOGRAPHY

[Ad] R.A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.

[Al1] W. Allegretto, *On positive L^∞ solutions of a class of elliptic systems*, Math Z. **191** (1986), 479-484.

[Al2] W. Allegretto, *Principal eigenvalues for indefinite-weight elliptic problems in $\mathbb{R}^n$*, Proc. Amer. Math. Soc., to appear.

[Am] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620-709.

[ANZ] W. Allegretto, P. Nistri and P. Zecca, *Positive solutions of elliptic non-positone problems*, Differential and Integral Equations, **5** (1), (1992), 95-101.

[AP] F.V. Atkinson and L.A. Peletier, *Ground states of $-\Delta u = f(u)$ and the Emden-Fowler Equation*, Arch. Rat. Mech. Anal. **93** (1986), 103-127.

[AR] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 369-381.

[AY] W. Allegretto and L.S. Yu, *Positive L^P-solutions of subcritical nonlinear problems*, J. Differential Equations, **87** (1990), 340-352.

[Be] M. Berger, *On the existence and structure of stationary states for a nonlinear Klein-Gordon equation*, J. Funct. Anal. **9** (1972), 249-261.

[BCF] K.J. Brown, C. Cosner and J. Fleckinger, *Principal eigenvalues for problems with indefinite weight function on $\mathbb{R}^n$*, Proc. Amer. Math. Soc. **109**, No. 1 (1990), 147-155.

[BCS] K.J. Brown, A. Castro and R. Shivaji, *Non-existence of radially symmetric non-negative solutions for a class of semi-positone problems*, Differential and Integral Equations , **2** (1989), 541-545.

[BF1] V. Benci and D. Fortunato, *Some compact embedding theorems for weighted Sobolev spaces*, Boll. U.M.I. **13-B** (1976), 832-843.

[BF2] V. Benci and D. Fortunato, *Second order elliptic operators on unbounded domains*, Boll. Un. Mat. Ital. **15-B** (1978), 193-209.

[BF3] V. Benci and D. Fortunato, *Weighted Sobolev spaces and the nonlinear Dirichlet problem in unbounded domains,* Ann. Mat. Pura. Appl. Series **4**, 121-122 (1979), 319-336.

[BK] H. Brezis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures et Appl. **58** (1979), 137-151.

[BL] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations I, II*, Arch. Rational Mech. Anal. **82** (1983), 313-345, 347-375.

[BL2] H. Berestycki and P.L. Lions, *Existence of stationary states of non-linear scalar field equations*, In: "Bifurcation Phenomena in Mathematical Physics and Related Topics," Bardos, C., Bessis, D. (eds.) Proc. NATO ASI, Cargese, 1979, Reidel, 1980, 269-292.

[BrL] H. Brezis and E.H. Lieb, *Minimum action solutions of some vector field equations,* Commun. Math. Phys. **96** (1984), 97-113.

[BS] M.S. Berger and M. Schechter, *Embedding theorems and quasilinear elliptic boundary value problems for unbounded domains*, Trans. Amer. Math. Soc. **172** (1972), 261-278.

[BSh] K.J. Brown and R. Shivaji, *Instability of nonnegative solutions for a class of semipositone problems,* Proc. Amer. Math. Soc., **112** (1991), 121-124.

[BT] H. Brezis and R.E.L. Turner, *On a class of superlinear elliptic problems,* Comm. in PDE **2** (1977), 601-614.

[Co] C. Cosner, *Positive solutions for superlinear elliptic systems without variational structure,* Nonlinear Analysis **8** (12) (1984), 1427-1436.

[CM] C.V. Coffman and M. Marcus, *Superlinear elliptic Dirichlet problems in almost spherically symmetric exterior domains*, Arch. Rational Mech. Anal. **96** (1986), 167-197.

[CS1] A. Castro and R. Shivaji, *Nonnegative solutions for a class of nonpositone problems,* Proc. Roy. Soc. Edinburgh, **108A** (1988), 291-302.

[CS2] A. Castro and R. Shivaji, *Nonnegative solutions for a class of radially symmetric nonpositone problems,* Proc. Amer. Math. Soc., **106** (1989), 735-740.

[CV] A. Chaljub-Simon and P. Volkmann, *Existence of ground states with exponential decay for semi-linear elliptic equations in* $\mathbb{R}^n$, J. Differential Equations **76** (1988), 374-390.

[De] K. Deimling, "Nonlinear Functional Analysis," Berlin/Heidelberg/New York/ Tokyo: Springer-Verlag, 1984.

[DN1] W.-Y. Ding and W.-M. Ni, *On the elliptic equation* $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ *and related topics,* Duke Math. J. **52** (1985), 485-506.

[DN2] W.-Y. Ding and W.-M. Ni, *On the existence of positive entire solutions of a semi-linear elliptic equation,* Arch. Rational Mech. Anal. **91** (1986), 283-308.

[Eg] H. Egnell, *Asymptotic results for finite energy solutions of semilinear elliptic equations,* J. Differential Equations **98** (1992), 34-56.

[Fo1] D. Fortunato, *On the index of elliptic partial differential operators in* $\mathbb{R}^n$, Ann. Mat. Pura. Appl. (4) **119** (1979), 317-331.

[Fo2] D. Fortunato, *Remarks on some nonlinear boundary value problems,* Ann. Mat. Pura Appl. **123-124** (1980), 217-231.

[Fu] N. Fukagai, *Existence and uniqueness of entire solutions of second order sublinear elliptic equations,* Funk. Ekvac. **29** (1986), 151-165.

[Fur1] Y. Furusho, *Existence of positive entire solutions for weakly coupled semilinear elliptic systems,* Proc. Roy. Soc. Edinburgh, Sect. A **120** (1992), 79-91.

[Fur2] Y. Furusho, *On decaying entire positive solutions of semilinear elliptic equations,* Japan J.Math. **14** (1989), 97-118.

[FLN] D.G. de Figuereido, P.L. Lions and R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations,* J. Math. Pures Appl. **61** (1982), 41-63.

[Gu] Y.-G. Gu, *Existence of nontrivial solutions of systems of elliptic equations on an unbounded domain,* J. Sys. Sci. and Math. Scis. **10** (2), (1990), 189-192.

[GL] D. Gogny and P.L. Lions, *Hartree-Fock theory in nuclear physics,* RAIRO Modél Math. et Anal., Num. **20** (1986), 571-638.

[GNN1] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243.

[GNN2] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Advances in Math. Stud. **A7** (1981), 369-402.

[GS] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations,* Comm. Partial Diff. Eq. **6** (1981), 883-901.

[GT] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," 2nd Edition, Springer-Verlag, Berlin/Heidelberg/New York/ Tokyo, 1983.

[Ka] N. Kawano, *On bounded entire solutions of semilinear elliptic equations*, Hiroshima Math. J. **14** (1984), 125-158.

[Kat] T. Kato, "Perturbation Theory for Linear Operators," 2nd Edition, Springer-Verlag, New York (1976).

[KK] N. Kawano and T. Kusano, *On positive entire solutions of a class of second order semilinear elliptic systems,* Math. Z. **186** (1984) 287-297.

[KN] T. Kusano and M. Naito, *Positive entire solutions of superlinear elliptic equations*, Hiroshima Math. J. **16** (1986), 361-366.

[KO] T. Kusano and S. Oharu, *On entire solutions of second order semilinear elliptic equations,* J. Math. Anal. Appl. **113** (1986), 123-135.

[KS] T. Kusano and C.A. Swanson, *A general method for quasilinear elliptic problems in* R^n, Journal Math. Anal. Appl. **167** (1992), 414-428.

[KSY] N. Kawano, J. Satsuma and S. Yotsutani, *Existence of positive entire solutions of an Emden-type elliptic equation*, Funkcialaj Eknacioj **31** (1988), 121-145.

[Li1] P.L. Lions, *On the existence of positive solutions of semilinear elliptic equations,* SIAM Rev. **24** (1982), 441-467.

[Li2] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The Locally compact case, I, II,* Ann. Inst. H. Poincaré Anal. Nonlineaire **1** (1984), 109-145, 223-283.

[Li3] P.L. Lions, *Solutions of Hartree-Fock equations for Couloub systems*, Comm. Math. Phys. **109** (1987), 33-97.

[LN1] Y. Li and W.-M. Ni, *On conformal scalar curvature equations in* $\mathbb{R}^n$, Duke Math. J. **57** (1988), 895-924.

[LN2] Y. Li and W.-M. Ni, *On the existence and symmetry properties of finite total mass solutions of the Matukuma equation, the Eddington equation and their generalizations*, Arch. Rat. Mech. Anal. **108** (1989), 175-194.

[LN3] Y. Li and W.-M. Ni, *On the asymptotic behaviour and radial symmetry of positive solutions of semilinear elliptic equations in* R^n, *I, II,* Arch. rational Mech. Anal. **118** (1992), 195-243.

[Ma] V.G. Maz'ja, "Sobolev Spaces," Springer-Verlag, Berlin/Heidelberg/ New York/Tokyo, 1985.

[Mi] C. Miranda, "Partial Differential Equations of Elliptic Type," Springer-Verlag, Berlin/New York, 1970.

[Ni1] W.-M. Ni, *On the elliptic equation* $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$, *its generalizations, applications to geometry,* Indiana Univ. Math. J. **31** (1982), 493-529.

[Ni2] W.-M. Ni, *Some aspects of semilinear elliptic equations on* $\mathbb{R}^n$, Nonlinear Diffusion Equations and their Equilibrium States, Vol 2 (1988), 171-205.

[Nu] R. Nussbaum, *Positive solutions of some nonlinear elliptic boundary value problems,* J. Math. Analysis Applic. **51** (1975), 461-482.

[NS1] E.S. Noussair and C.A. Swanson, *Positive solutions of elliptic systems with bounded nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 321-332.

[NS2] E.S. Noussair and C.A. Swanson, *Asymptotics for semilinear elliptic systems,* Canad. Math. Bull., Vol. **34** (4), (1991), 514-519.

[NS3] E.S. Noussair and C.A. Swanson, *Properties of potential systems in* $\mathbb{R}^N$, J. Differential Equations **95** (1992), No. 1, 1-19.

[NS4] E.S. Noussair and C.A. Swanson, *An* $L^q(\mathbb{R}^n)$*-theory of subcritical semilinear elliptic problems,* J. Differential Equations **84** (1990), 52-61.

[NS5] E.S. Noussair and C.A. Swanson, *Positive $L^q(\mathbb{R}^n)$-solutions of subcritical Emden-Fowler problems,* Arch. Rational Mech. Anal. **101** (1988), 85-93.

[NS6] E.S. Noussair and C.A. Swanson, *Positive solutions of semilinear elliptic problems in unbounded domains,* J. Diff. Equations **57** (1985), 349-372.

[NS7] E.S. Noussair and C.A. Swanson, *Positive decaying entire solutions of superlinear elliptic equations*, Indiana Univ. Math. J. **36** (1987), 651-657.

[Pi] Y. Pichover, *On the equivalence of Green functions of second order elliptic equations in* $\mathbb{R}^n$, Differential and Integral Equations **5**, No. 3 (1992), 481-493.

[Po] S.I. Pohozaev, *Eigenfunctions for the equations* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **6** (1965), 1408-1411.

[Qi] J. Qing, *A priori estimates for positive solutions of semilinear elliptic systems,* J. Partial Differential Eq. **1** (2), Ser. A (1988), 61-70.

[Ra] P.H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," Amer. Math. Soc., Providence, RI, 1986.

[Sc] M. Schechter, *Basic theory of Fredolm operators*, Ann. Scuola Norm. Sup. di Pisa et Scienze, **21** (1967), 261-280.

[Sh] A.W. Shaker, *On symmetry in elliptic systems,* Applicable Analysis, Vol. **41** (1991), 1-9.

[St] G. Stampacchia, *Equations elliptiques du second order à coefficients discontinues,* Les Presses de l'Univ. de Montréal (1966).

[Str] M. Struwe, "Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems," Springer-Verlag, Berlin/ Heidelberg/New York, 1990.

[SW1] J. Smoller and A. Wasserman, *Symmetry-breaking for semilinear elliptic equations with general boundary conditions,* Comm. Math. Phys., **105** (1986), 415-441.

[SW2] J. Smoller and A. Wasserman, *Symmetry-breaking for positive solutions of semilinear elliptic equations,* Arch. Rational Mech. Anal., **95** (1986), 217-225.

[SW3] J. Smoller and A. Wasserman, *Existence of positive solutions for semilinear elliptic equations in general domains,* Arch. Rational Mech. Anal., **98** (1987), 229-249.

[Tr] W.C. Troy, *Symmetry properties in systems of semilinear elliptic equations,* J. Differential Equations **42** (1981), 400-413.

[Un] S. Unsurangsie, *Existence of a solution for a wave equation and elliptic problems,* Ph.D. Thesis (1988), A. Castro, supervisor, University of North Texas.

[Wo] J.S.W. Wong, *On the generalized Emden-Fowler equation,* SIAM Rev. **17** (1975), 339-360.

[Ze] E. Zeidler, "Nonlinear Functional Analysis and its Applications," Vol. **I**, Springer-Verlag, New York, 1986.